Delays Part 2
Equilibrium Behaviour
Higher-Order Delays

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CMPT 858

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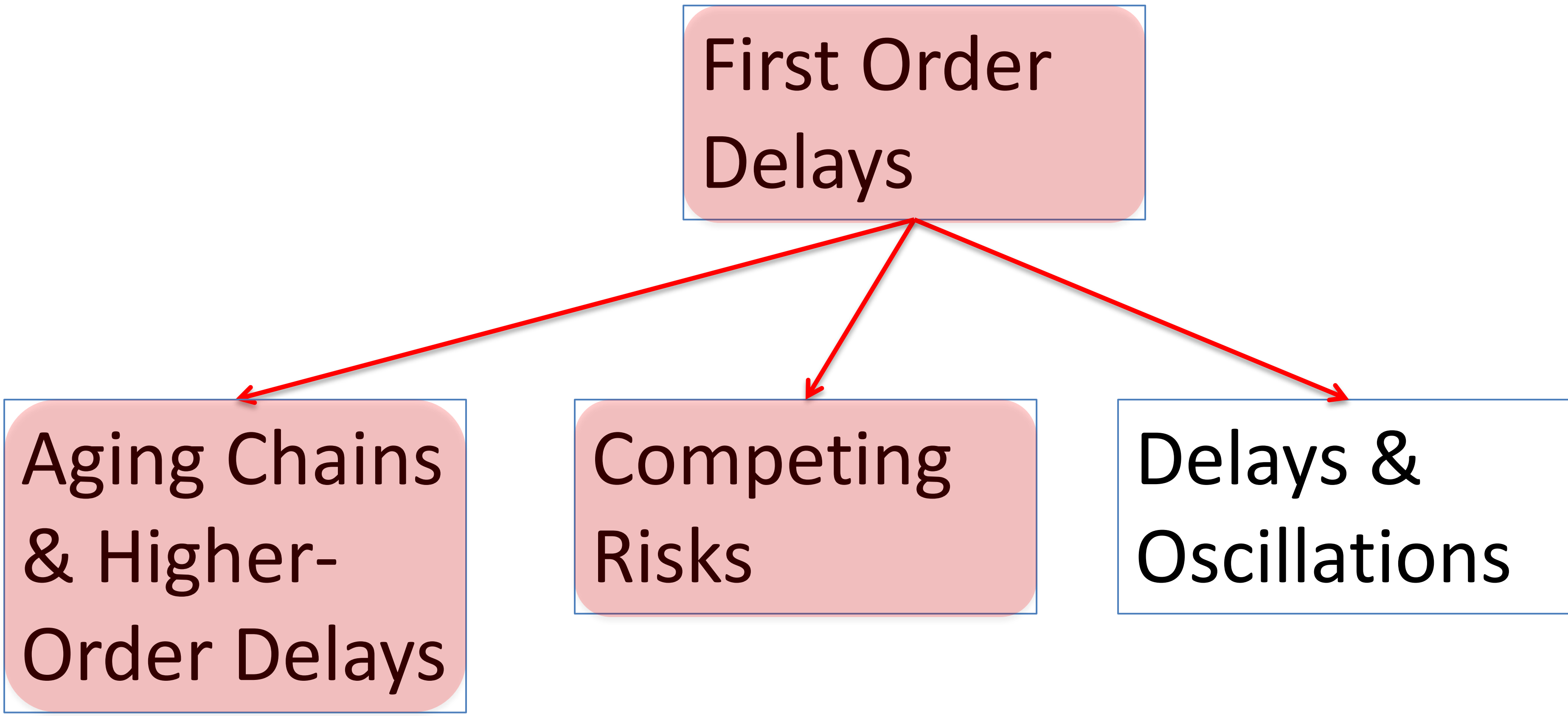
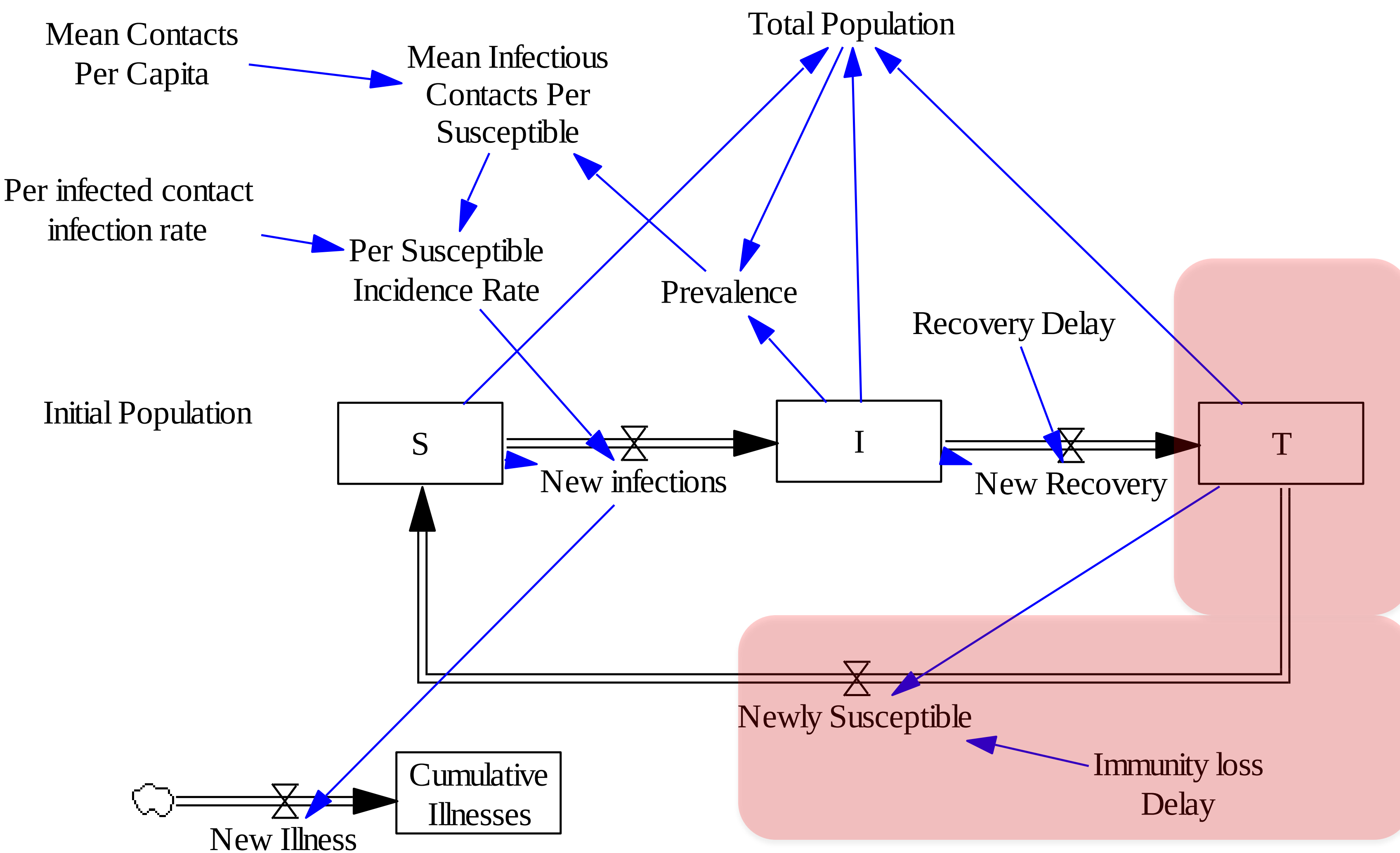
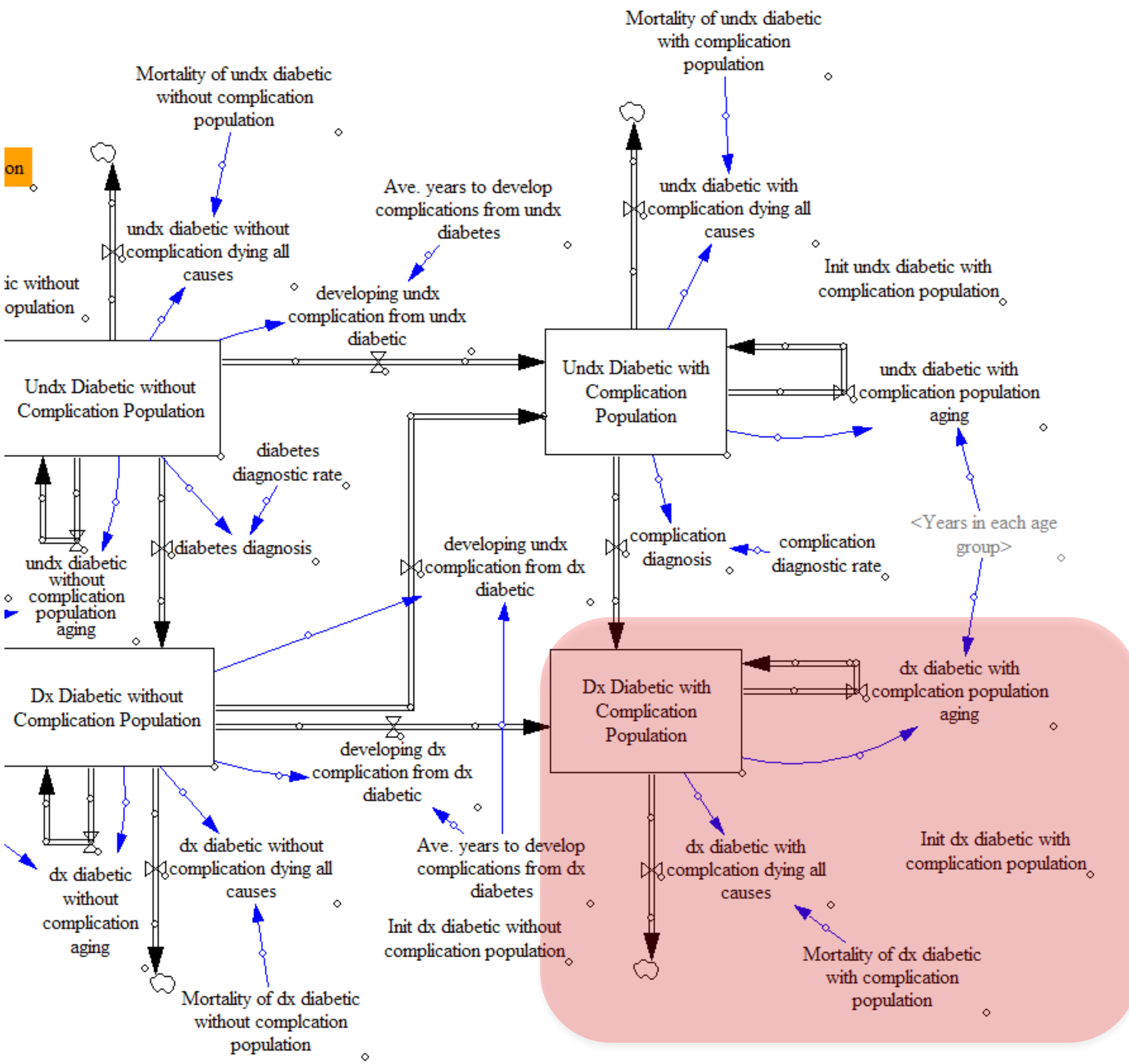
Our Route Forward:
3 Common Types of Delay-Related Dynamics



First Order Delays in Action: Simple SIT Model

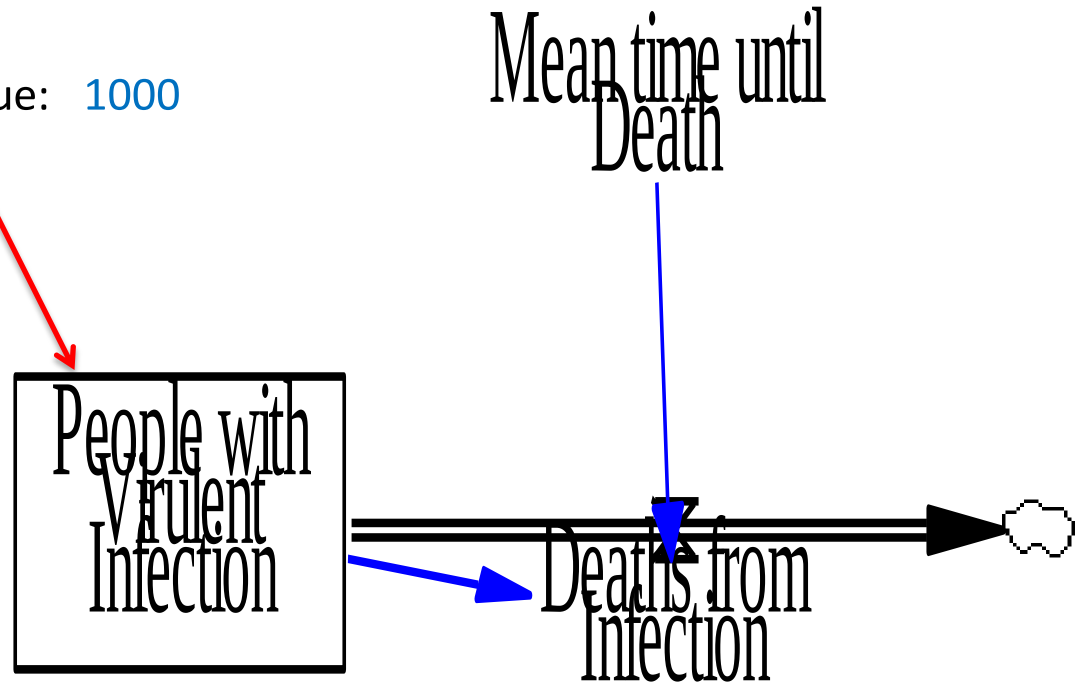


First Order Delays in Action: Simple SIT Model



Recall: Simple First-Order Decay

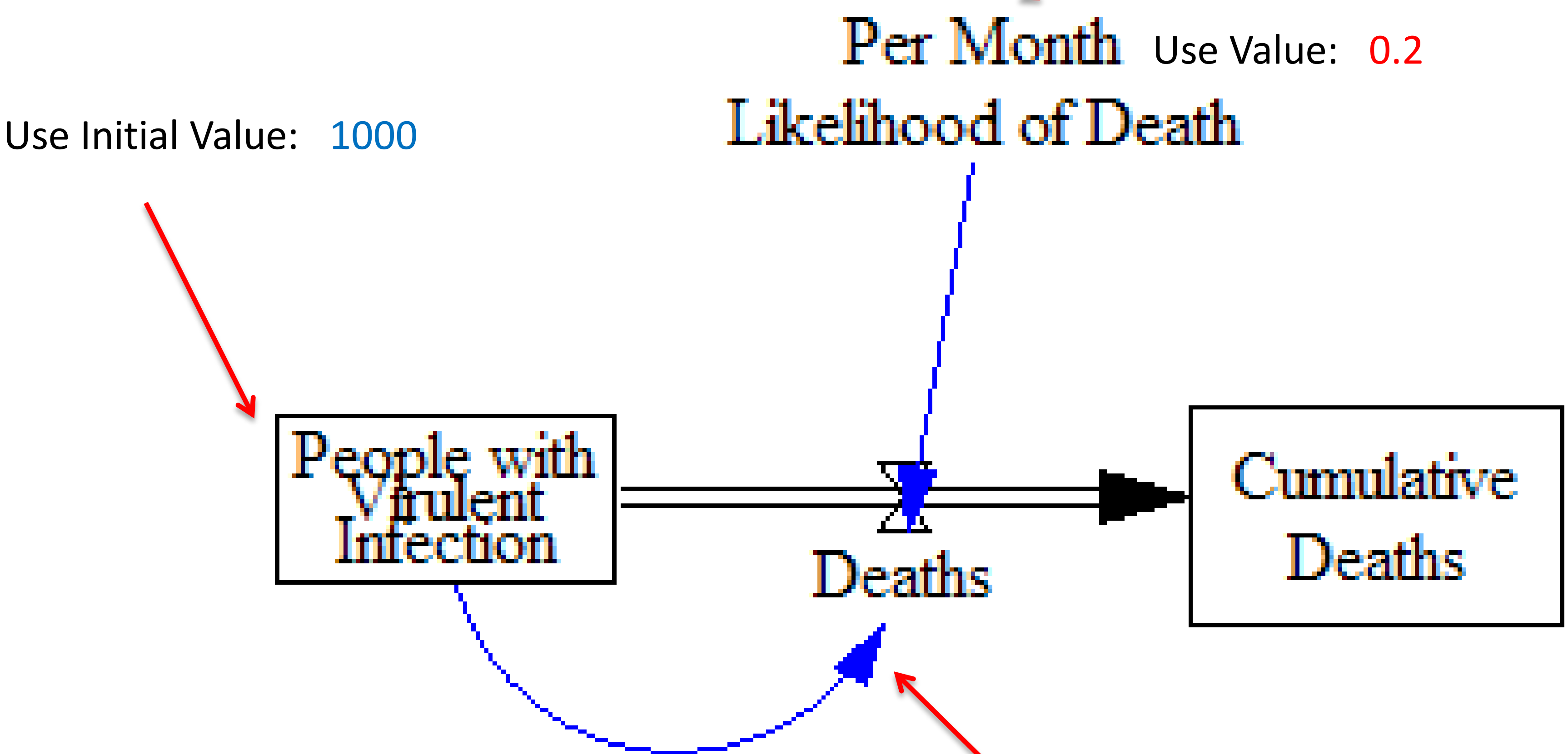
Use Initial Value: 1000



Use Formula: People with Virulent Infection/Mean time until Death

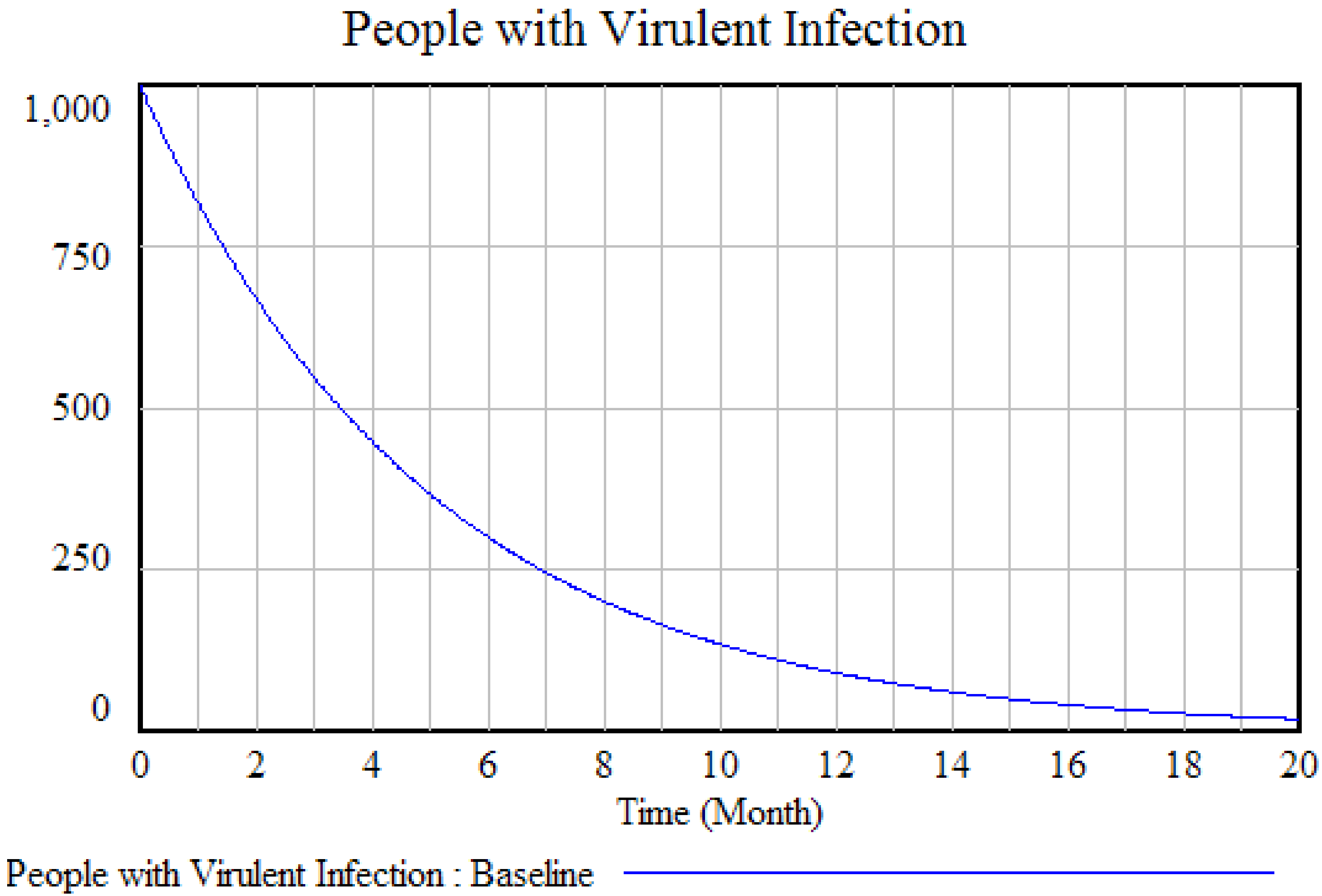
First-Order Decay (Variant of Last Time)

Recall: How does this relate to the mean time until death?

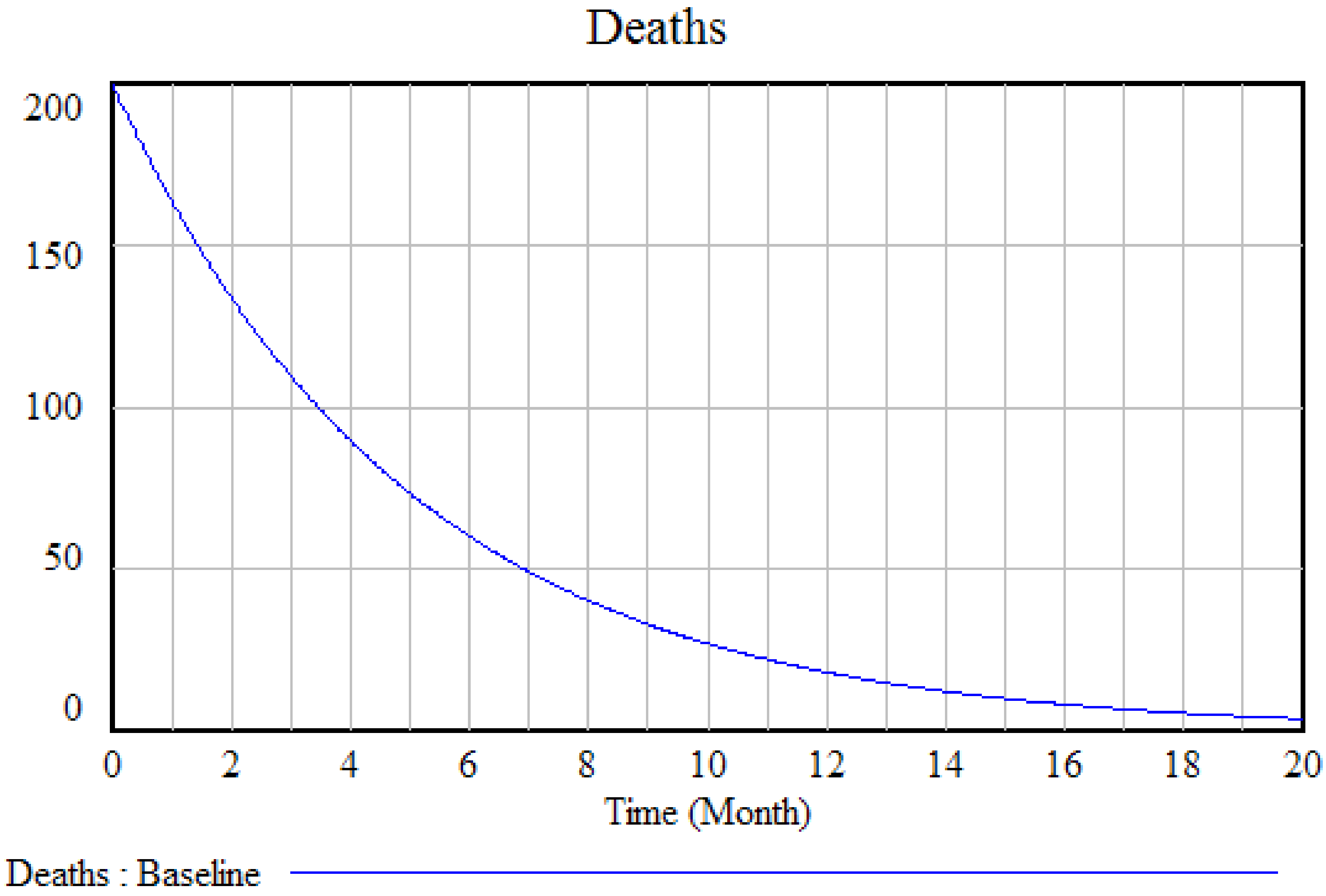


Use Formula: People with Virulent Infection*Per Month Likelihood of Death

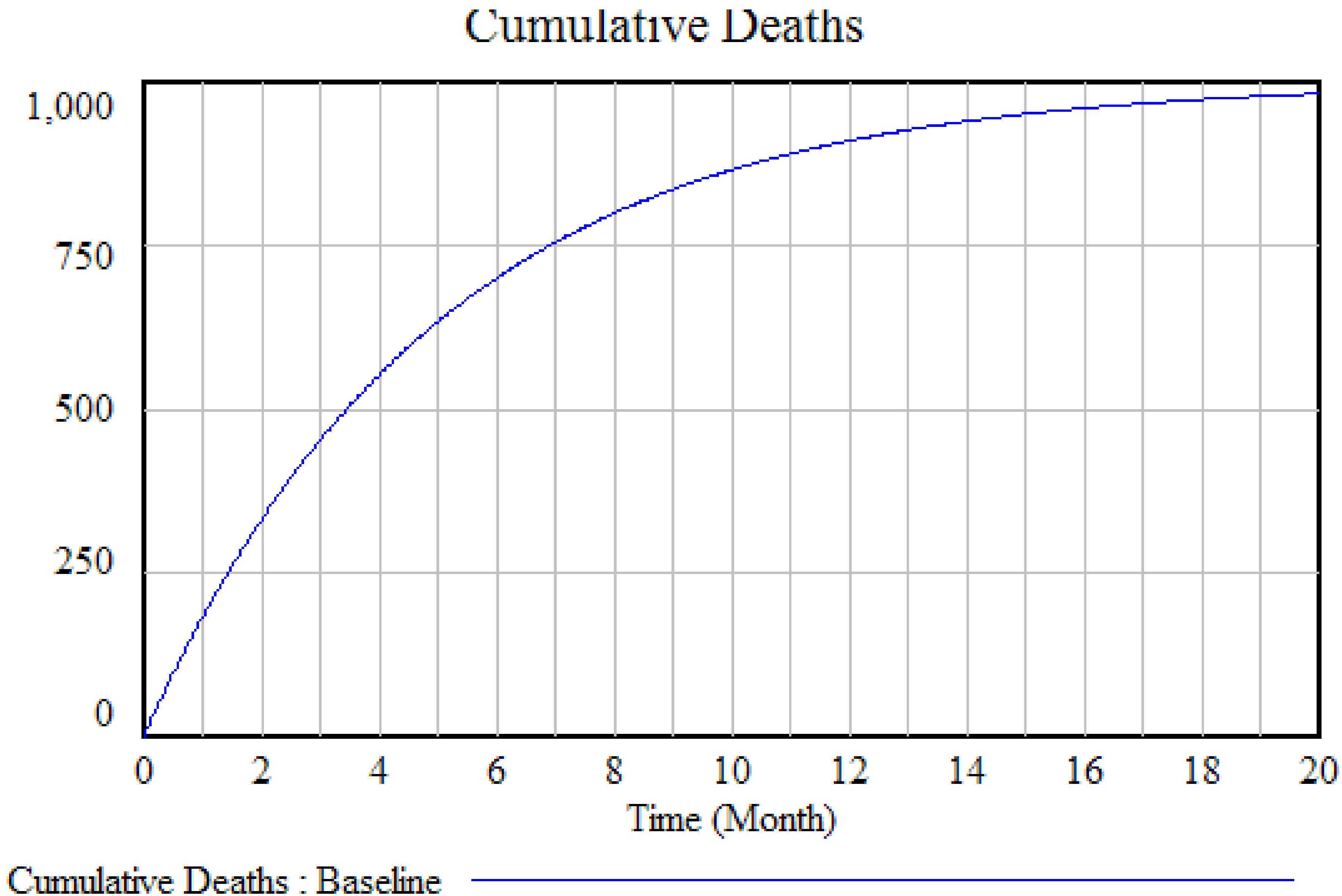
People in Stock



Flow Rate of Deaths



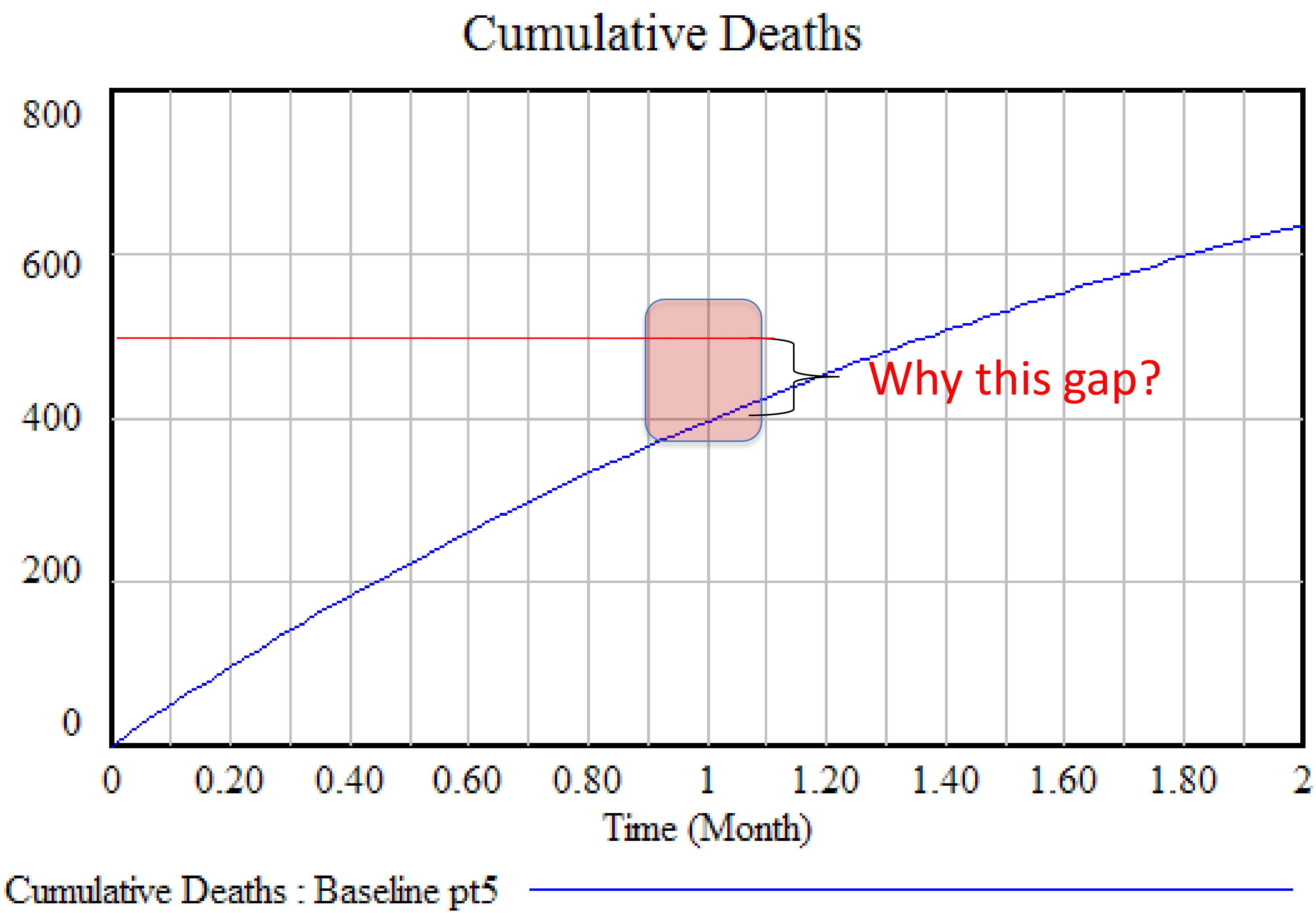
Cumulative Deaths

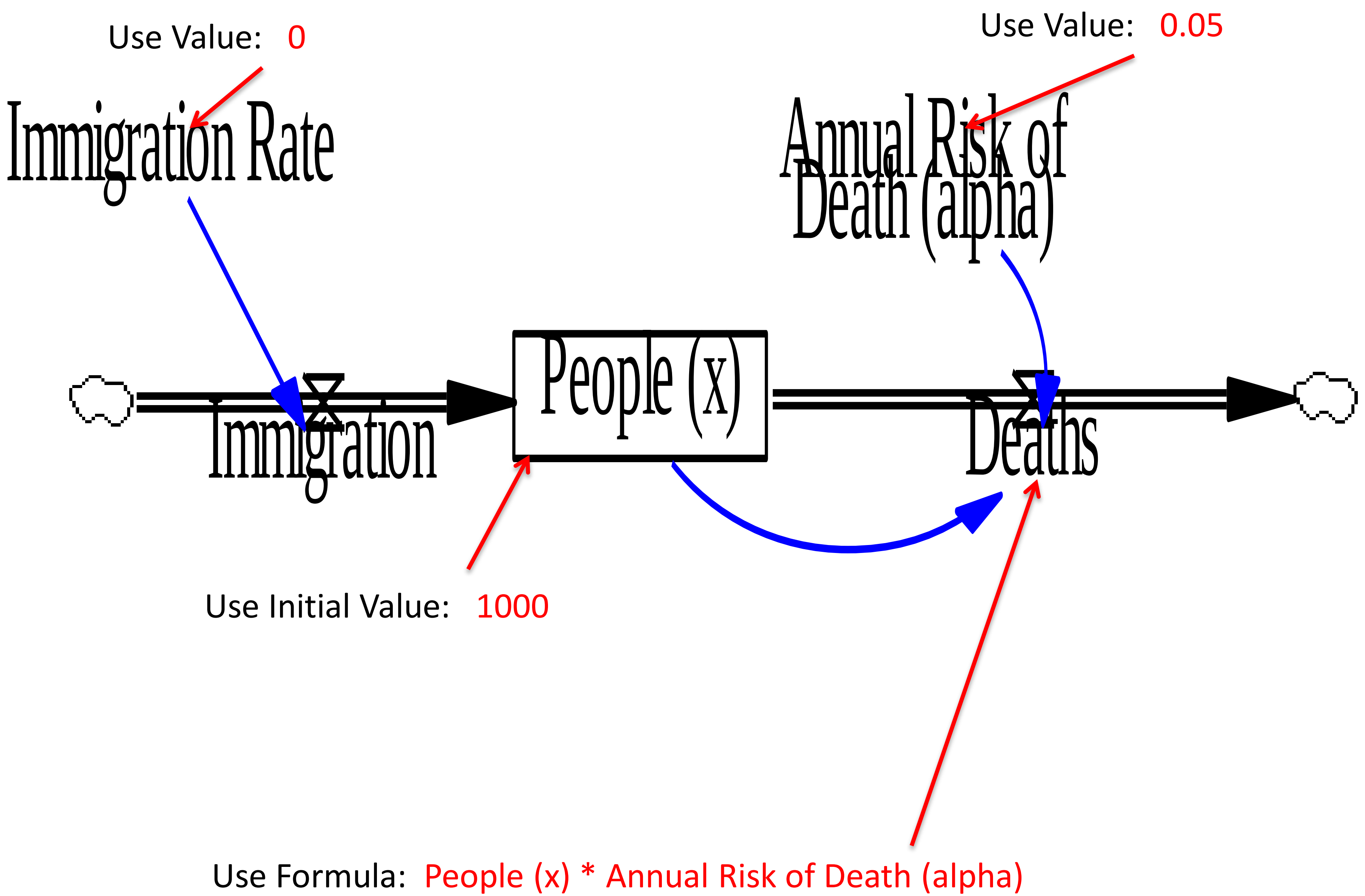


Closeup


Cumulative Deaths
400
300
200
100
0
0
0.20
0.40
0.60
0.80
1
1.20
1.40
1.60
1.80
2
Time (Month)
Cumulative Deaths : Baseline
Why this gap?

50% per Month Risk of Deaths



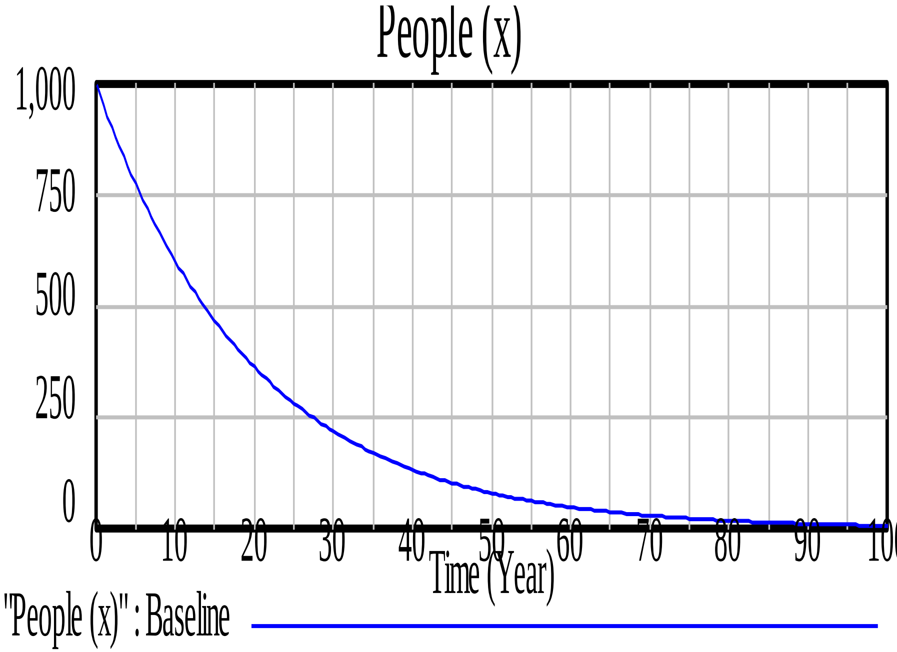

Use Value: 0
Immigration Rate
Use Value: 0.05
Annual Risk of Death (alpha)
Immigration
People (x)
Deaths
Use Initial Value: 1000
Use Formula: People (x) * Annual Risk of Death (alpha)

Questions

- What is behaviour of stock x?
- What is the mean time until people die?
- Suppose we had a constant inflow – what is the behaviour then?

Answers

- Behaviour Of Stock



- Mean Time Until Death

  Recall that if coefficient of first order delay is α , then mean time is $1/\alpha$ (Here, 1/0.05 = 20 years)

Equilibrium Value of a First-Order Delay

- Suppose we have flow of rate i into a stock with a first-order delay out
  - This could be from just a single flow, or many flows
- The value of the stock will approach an equilibrium where inflow=outflow

Equilibrium Value of 1st Order Delay

- Recall: Outflow rate for 1st order delay=αx
  - Note that this depends on the value of the stock!
- Inflow rate=i
- At equilibrium, the level of the stock must be such that inflow=outflow
  - For our case, we have

    αx=i

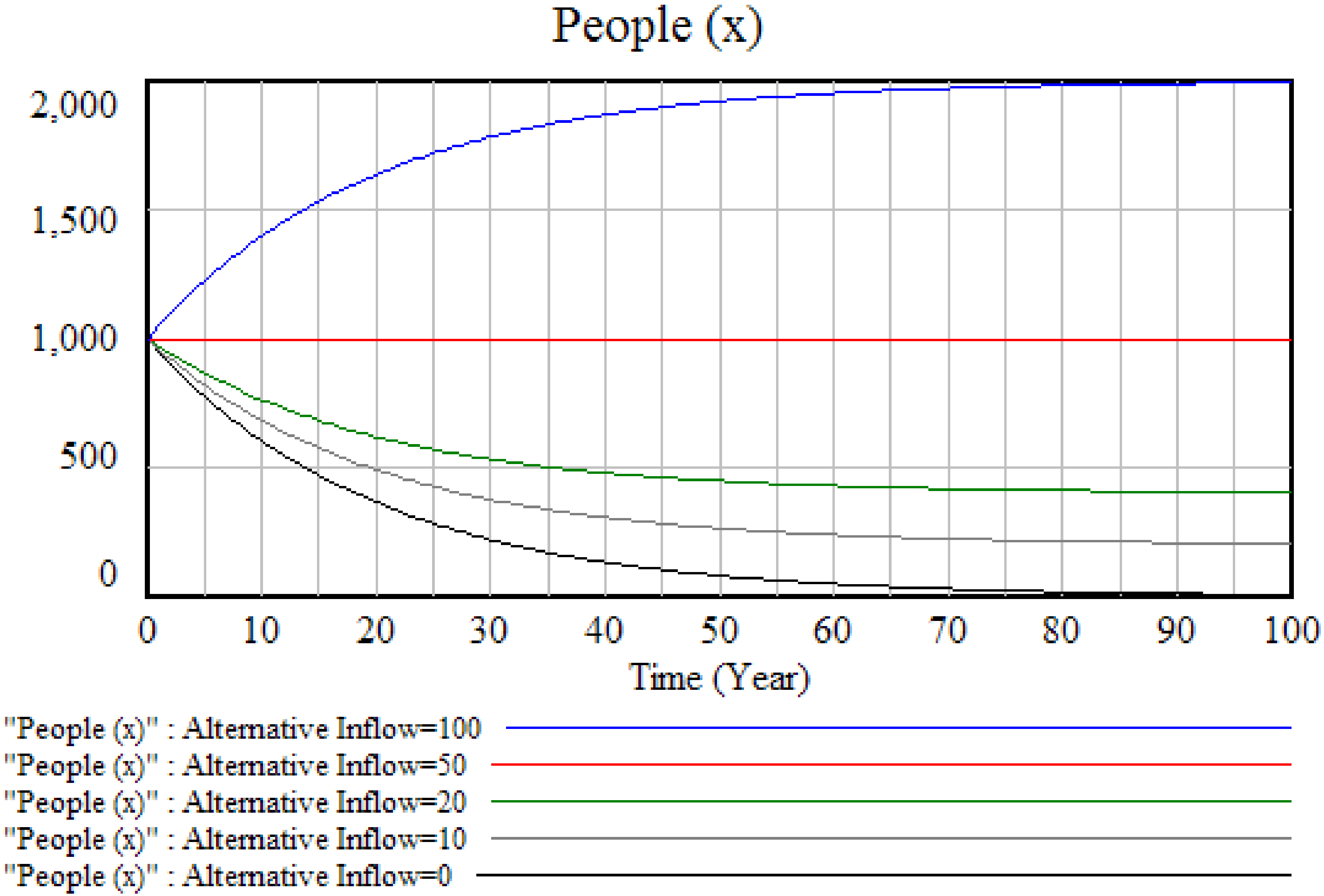
    Thus x=i/α

The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow

Scenarios for First Order Delay: Variation in Inflow Rates

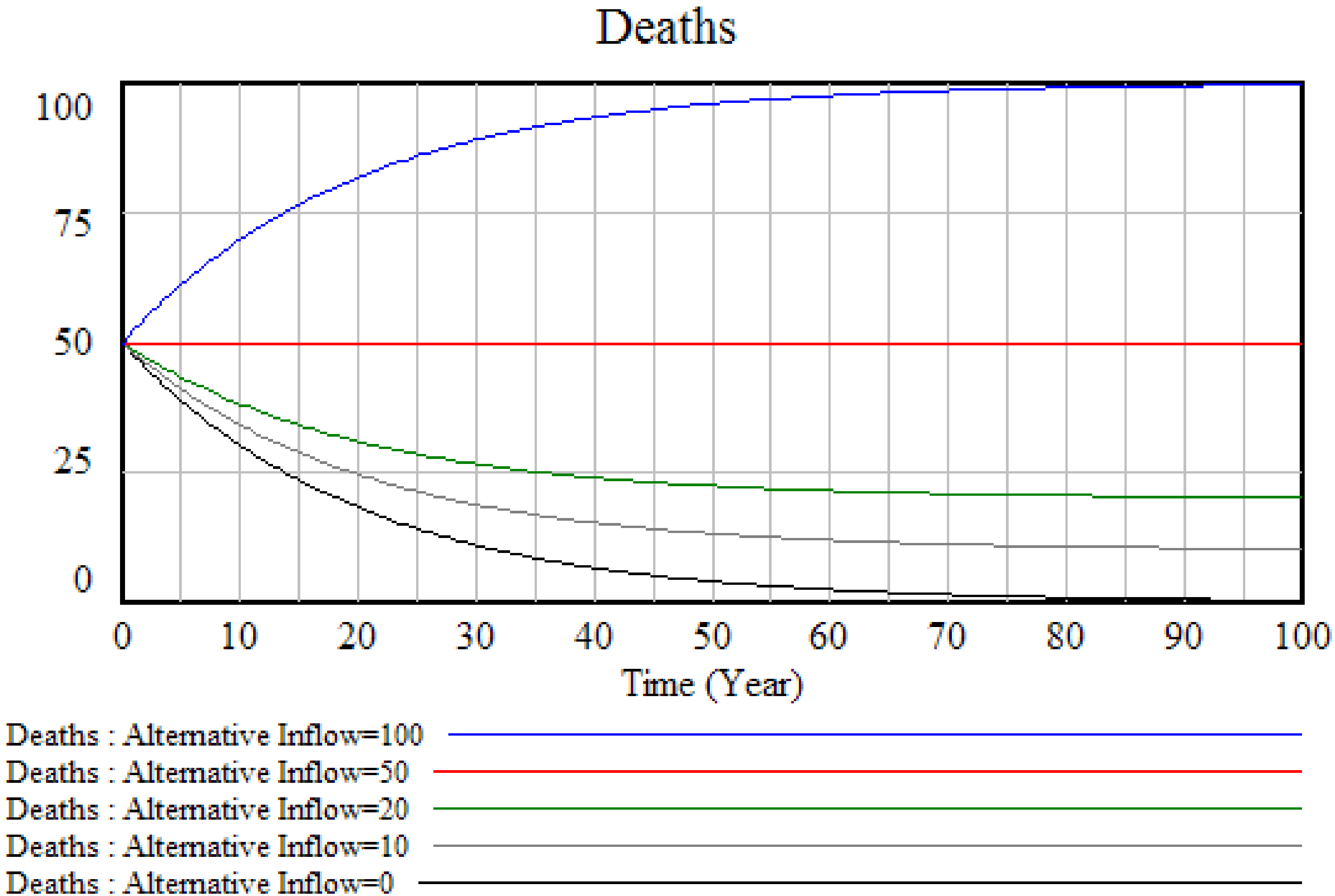
- For different immigration (inflows) (what do you expect?)
  - Inflow=10
  - Inflow=20
  - Inflow=50
  - Inflow=100
  - Why do you see this “goal seeking” pattern?
  - What is the “goal” being sought?

Behaviour of Stock for Different Inflows



Why do we see this behaviour?

Behaviour of *Outflow* for Different Inflows

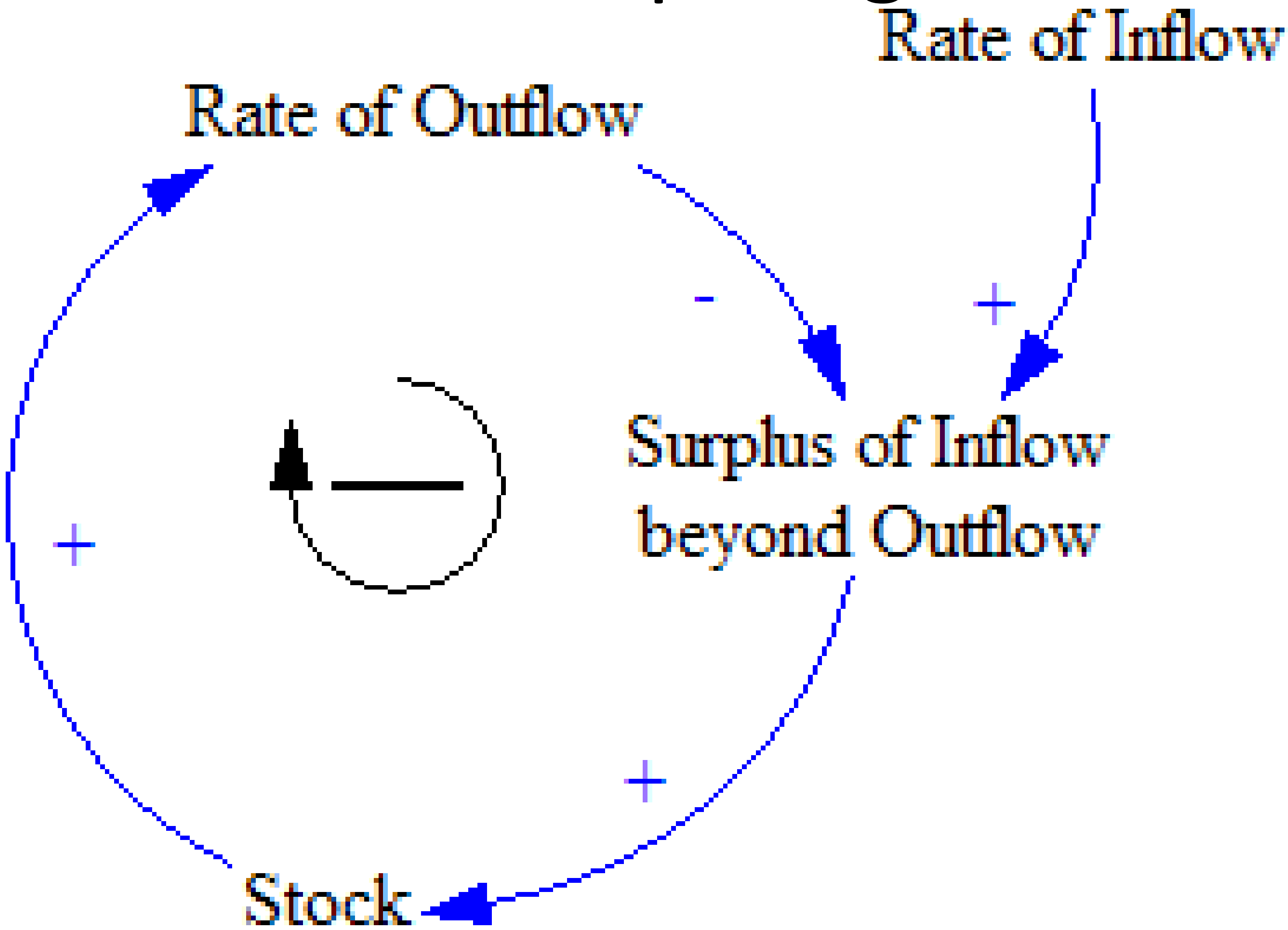


Why do we see this behaviour? Imbalance (gap) causes change to stock (rise or fall) ⇒ change to outflow to lower gap **until outflow=inflow**

Goal Seeking Behaviour

- The goal seeking behaviour is associated with a negative feedback loop
  - The larger the population in the stock, the more people die per year
- If we have more people coming in than are going out per year, the stock (and, hence, outflow!) rises until the point where inflow=outflows
- If we have fewer people coming in than are going out per year, the stock declines (& outflow) declines until the point where inflow=outflows

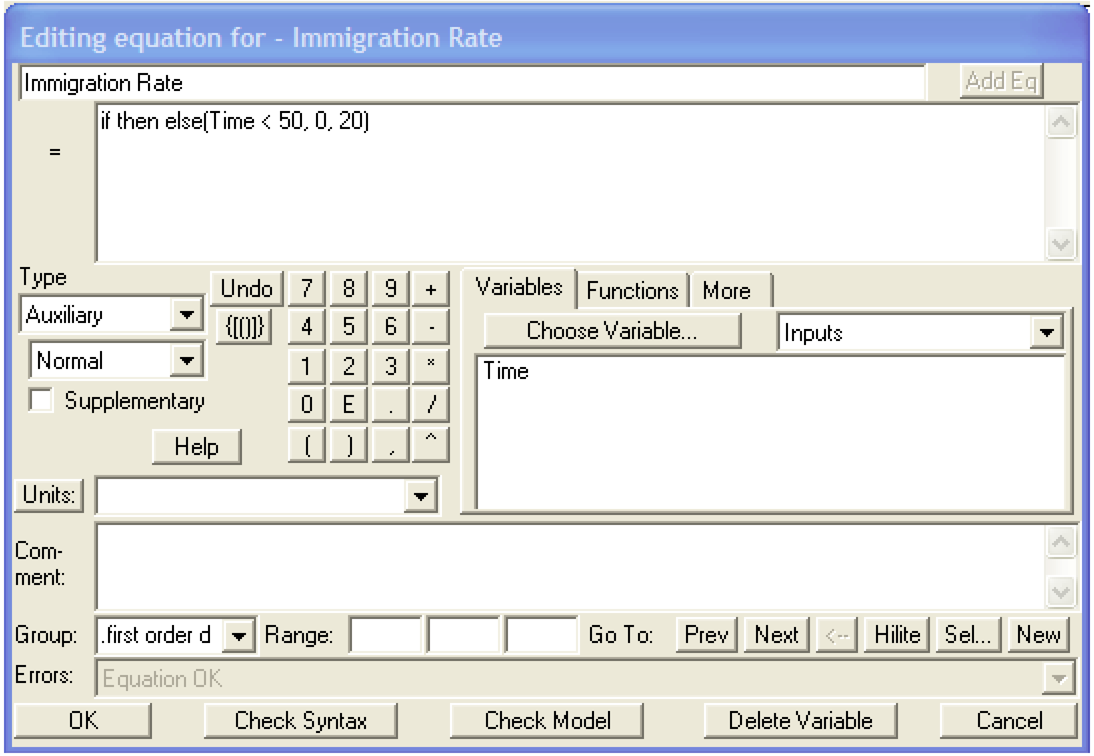
As a Causal Loop Diagram



What does this tell us about how the system would respond to a sudden change in immigration?

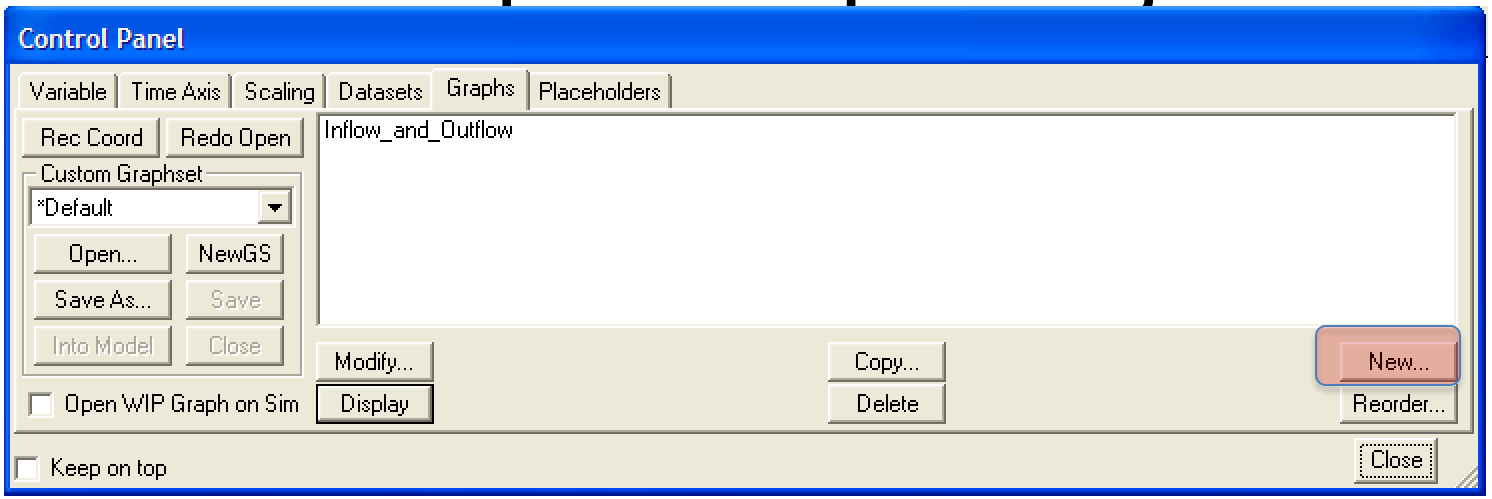
Response to a Change

- Feed in an immigration “step function” that rises suddenly from 0 to 20 at time 50

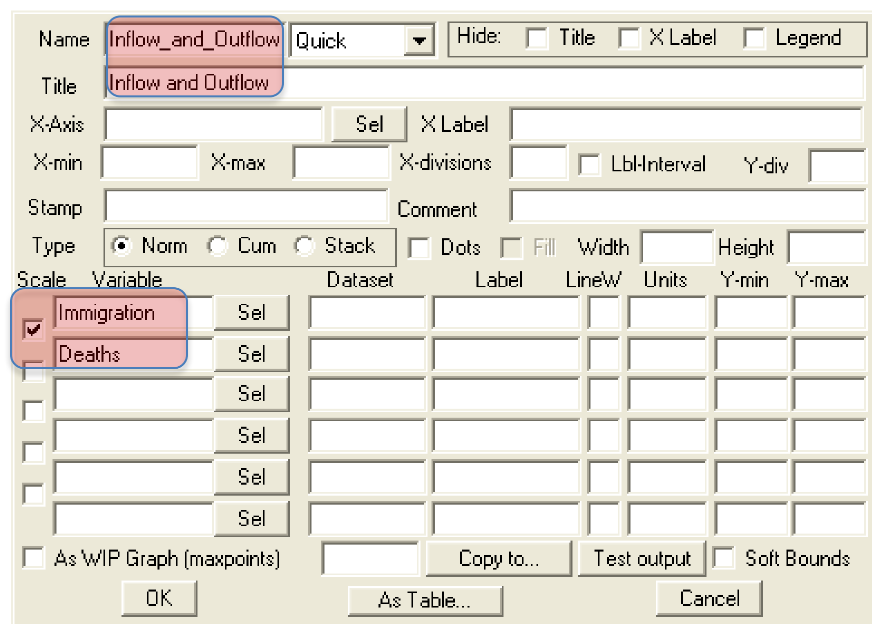


- Set the Initial Value of Stock to 0
- How does the stock change over time?

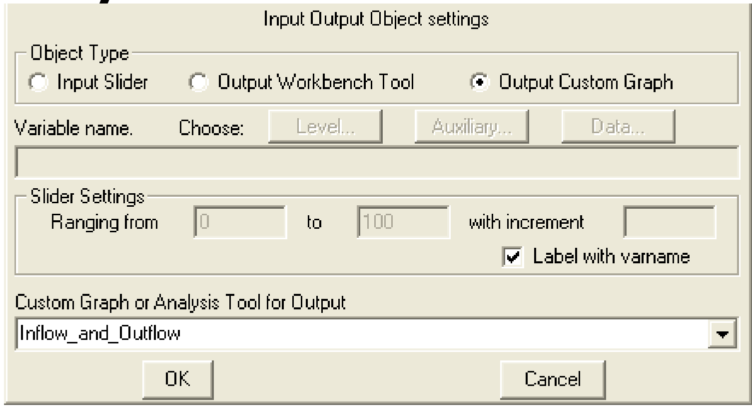
Create a Custom Graph & Display it as an Input-Output Object

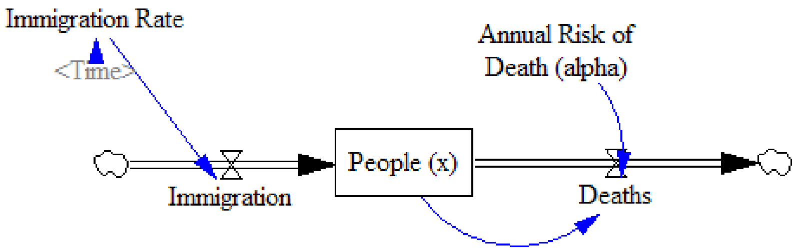


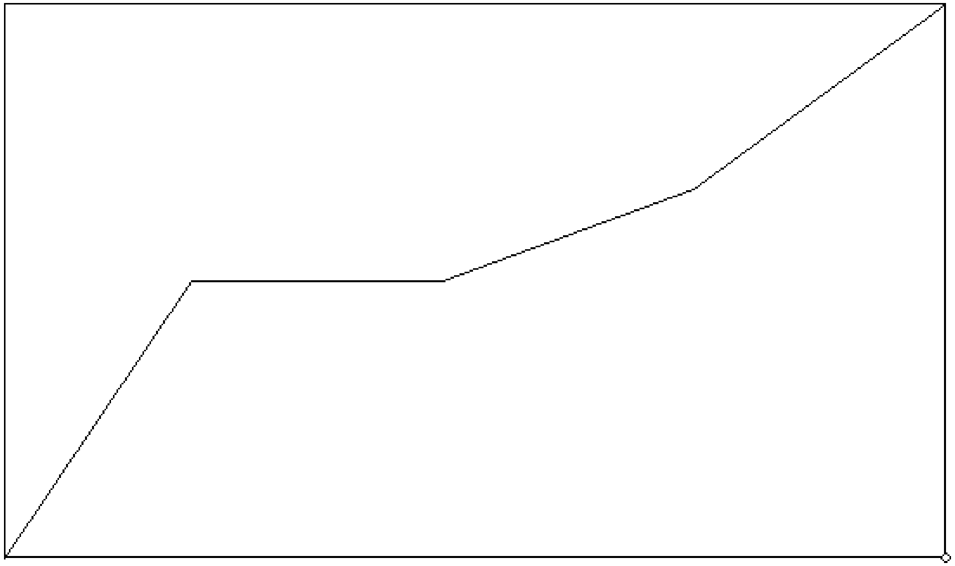
- Editing



Create Input-Output Object (for Synthesim)

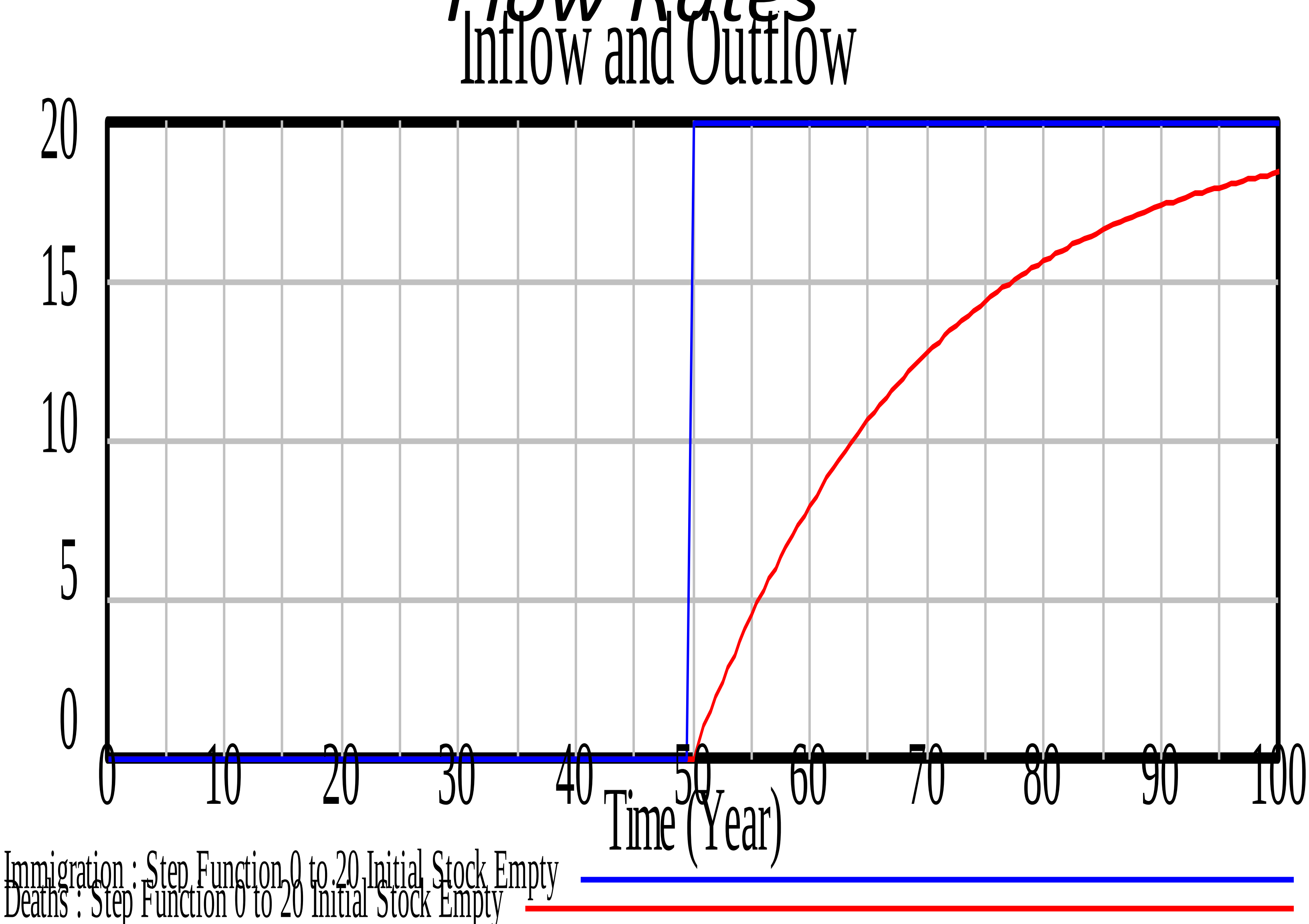






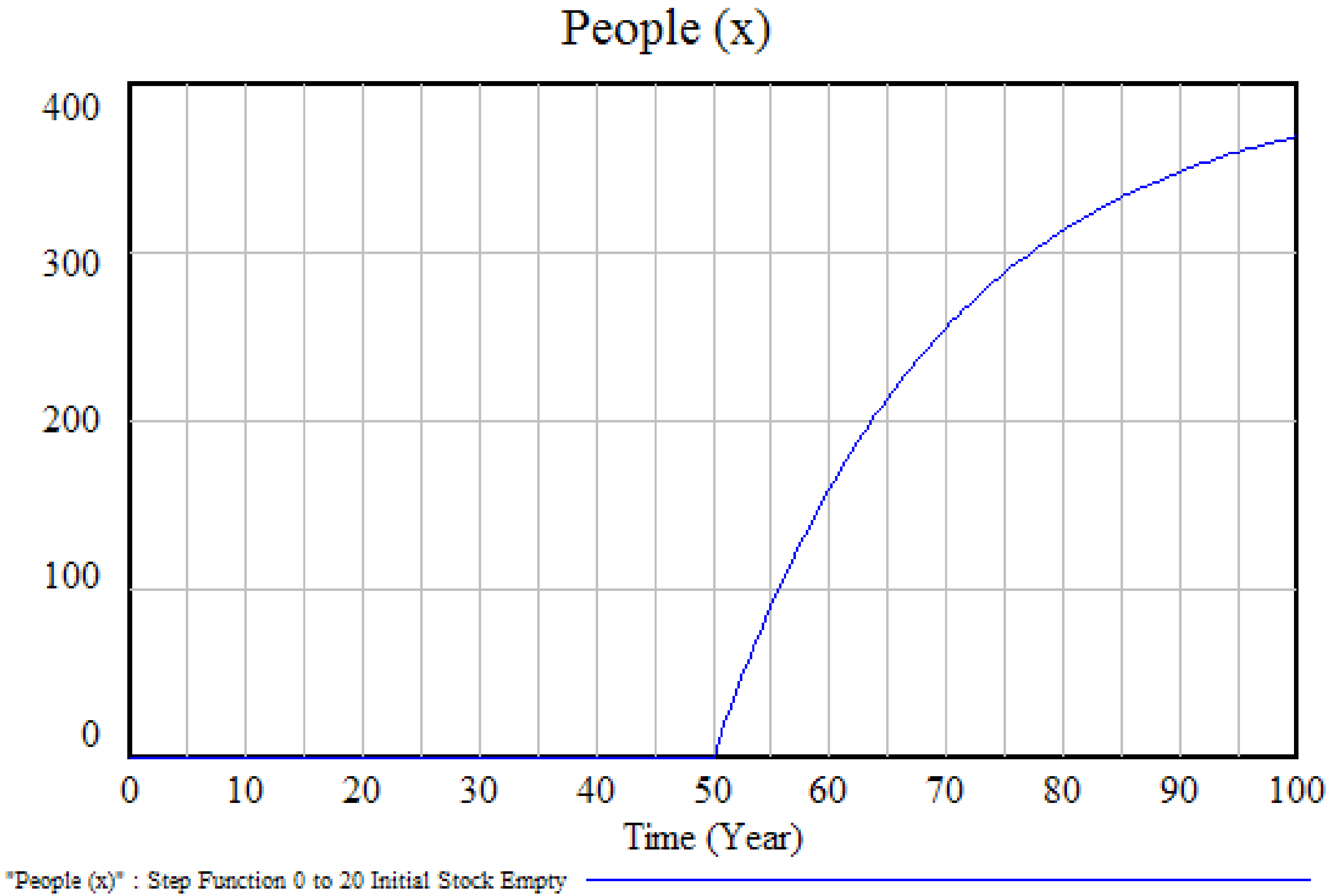
Stock Starting Empty

Flow Rates



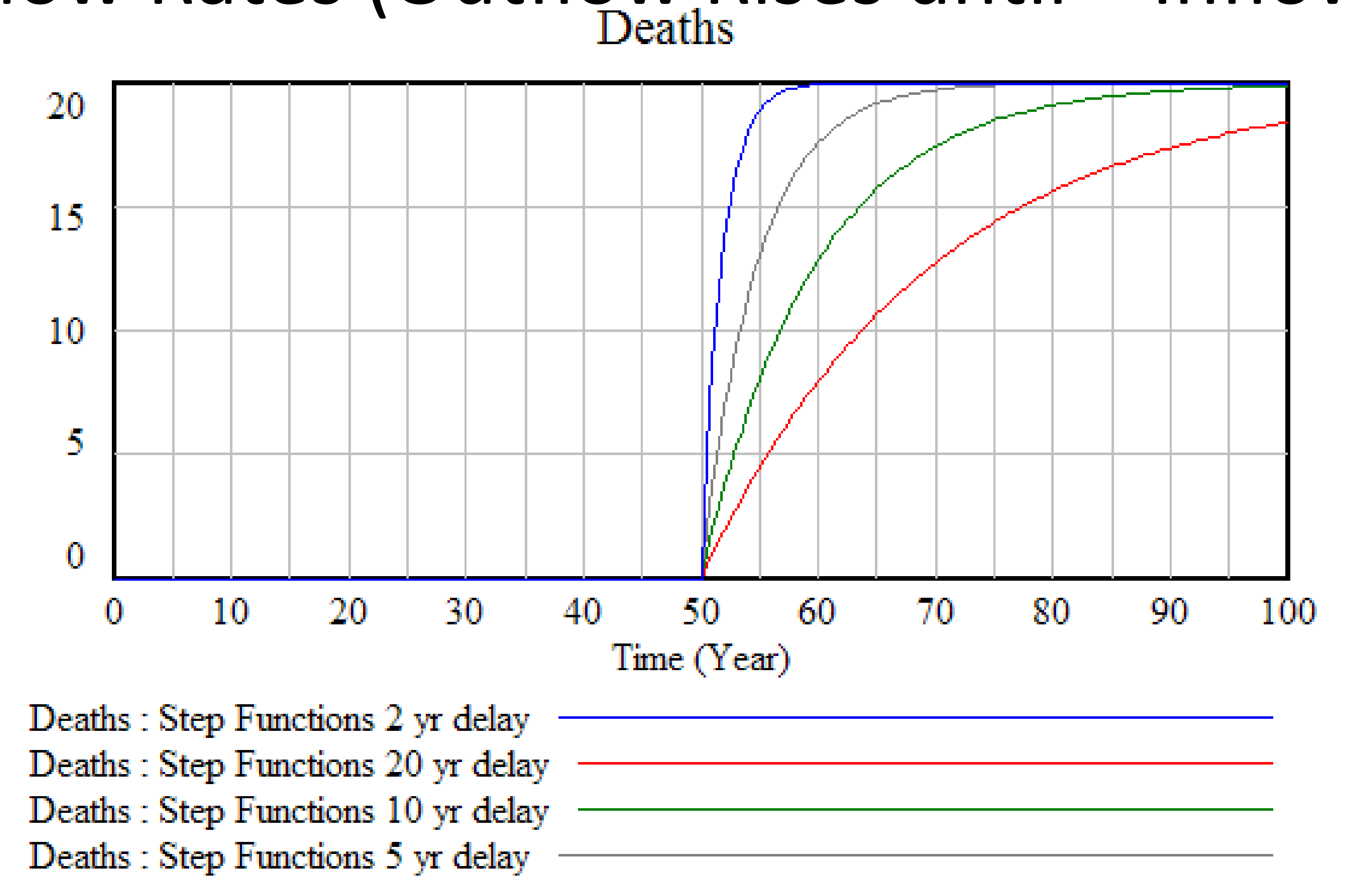
How would this change with alpha?

Stock Starting Empty?
Value of *Stock* (Alpha=.05)



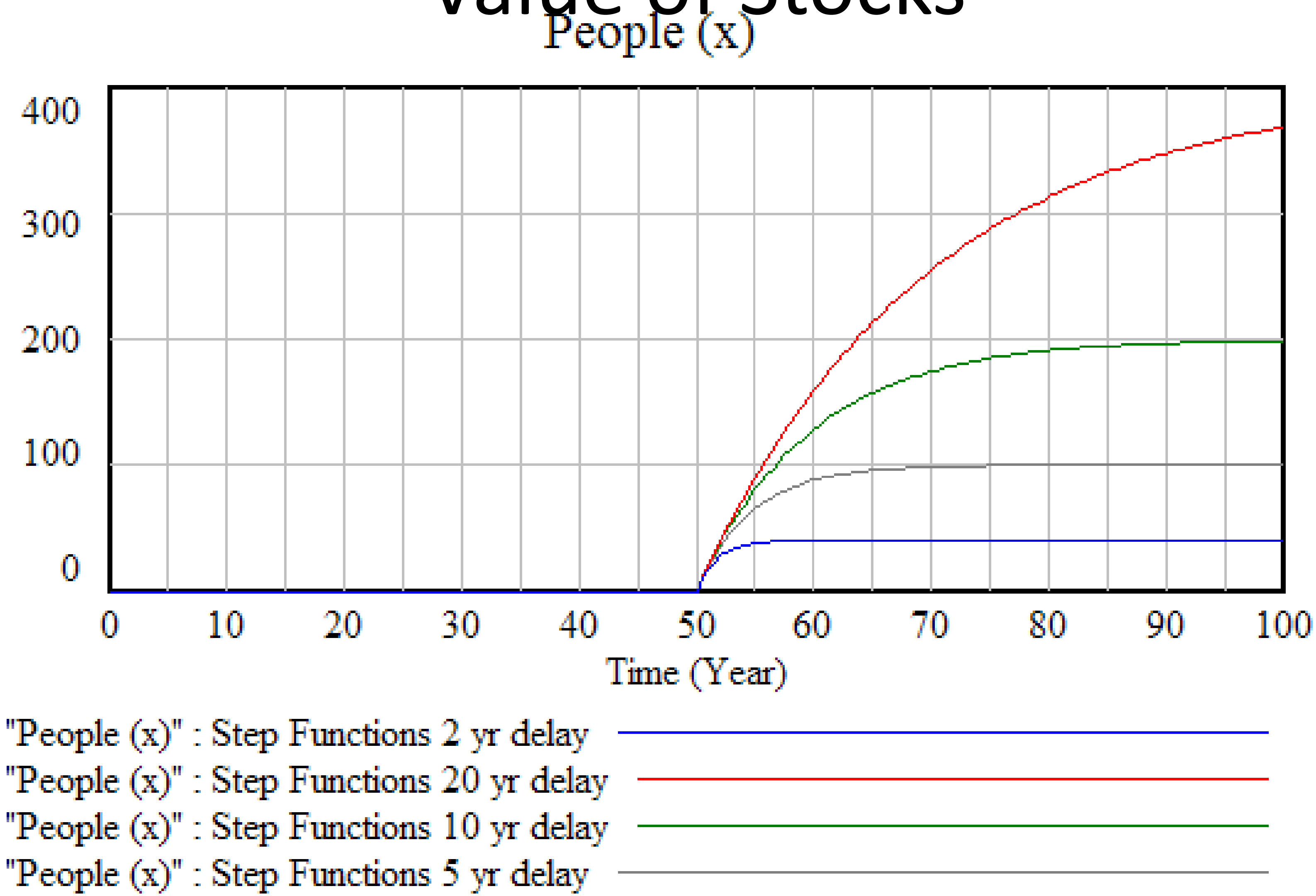
How would this change with alpha?

For Different Values of (1/) Alpha
Flow Rates (Outflow Rises until = Inflow)



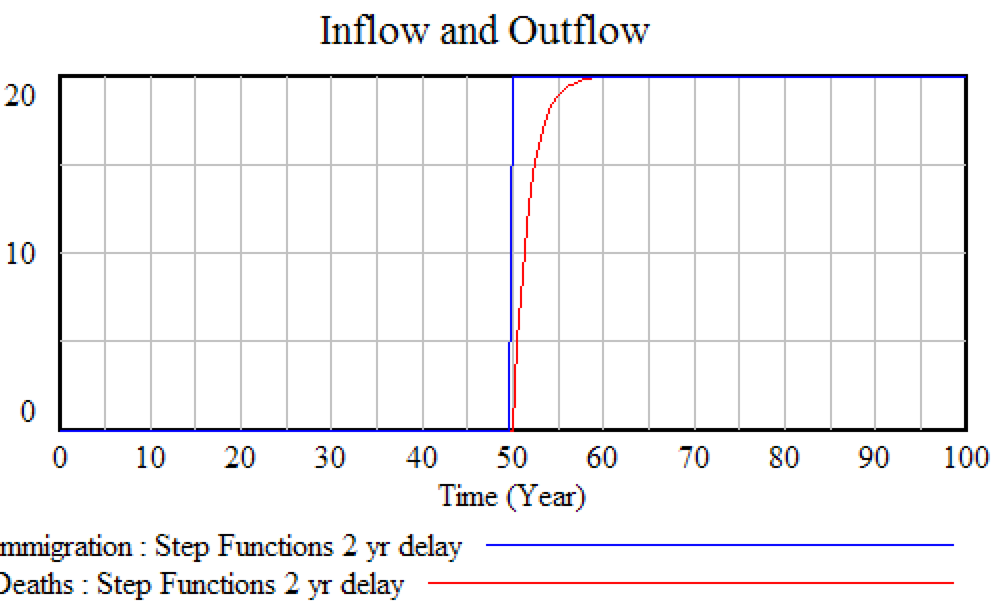
This is for the *flows*. What do stocks do?

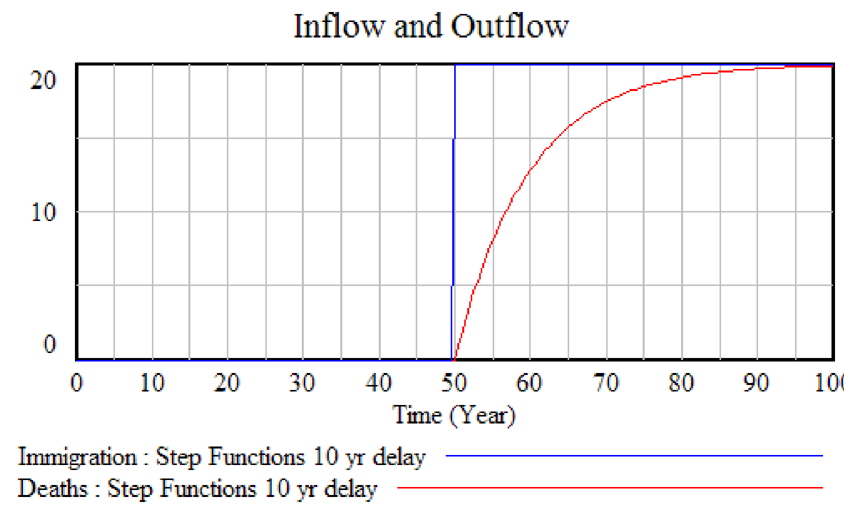
For Different Values of (1/) Alpha
Value of Stocks

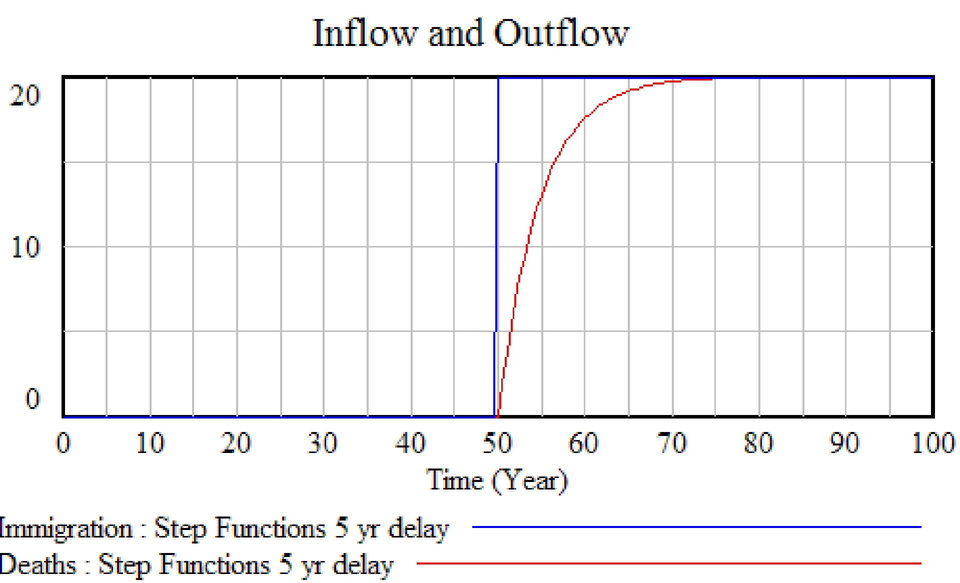


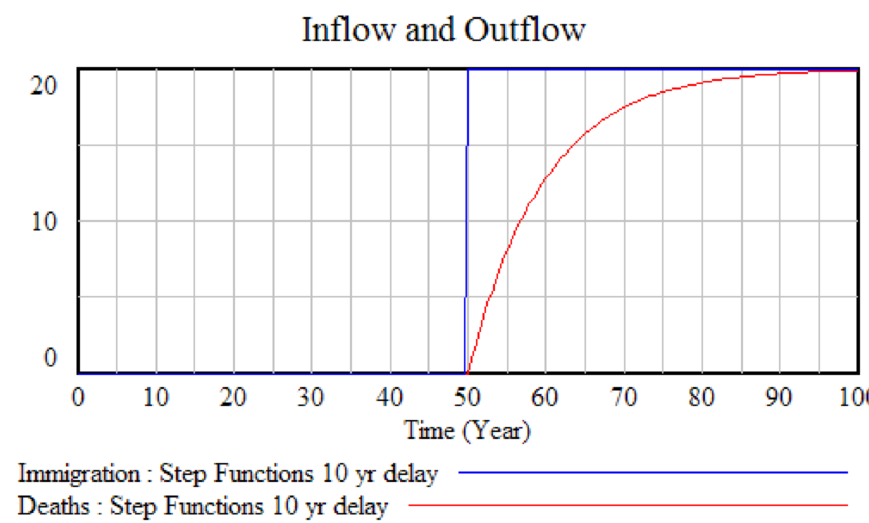
Why do we see this behaviour? A longer time delay (or smaller chance of leaving per unit time) requires *x* to be *larger* to make outflow=inflow

Outflows as Delayed Version of Inputs



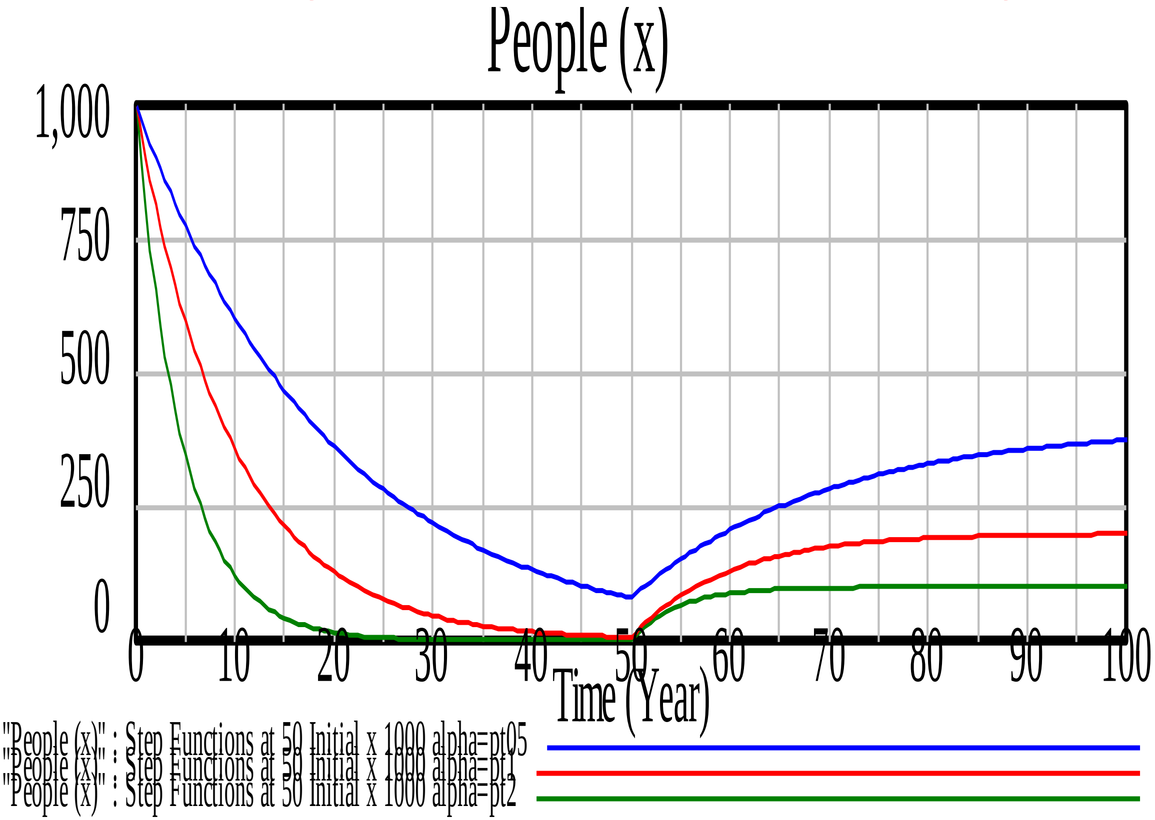


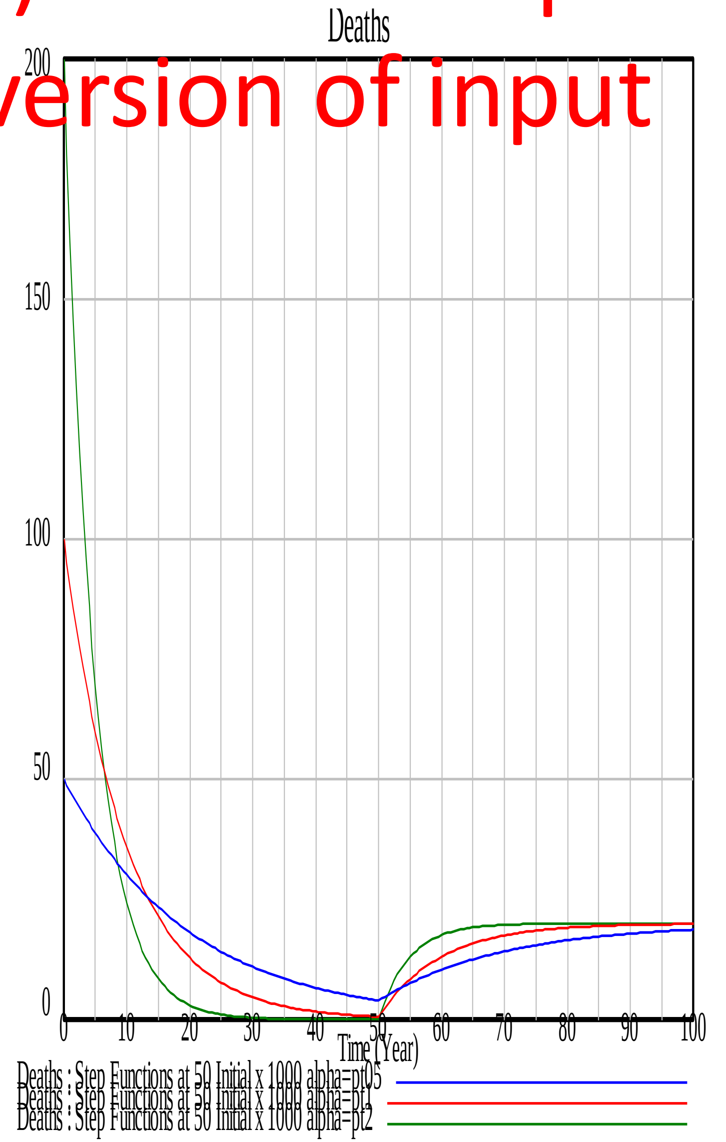




What if stock doesn't start empty?

Decays at first (no inflow) & then output responds with delayed version of input





Higher Order Delays & Aging Chains

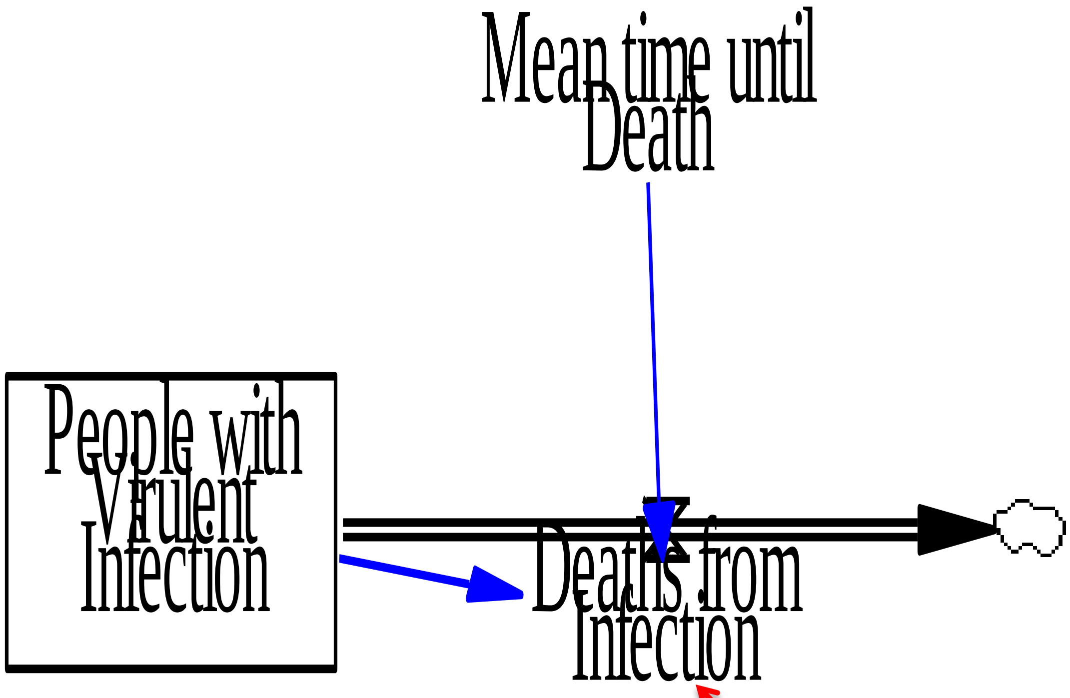
Moving Beyond the “memoryless assumption”

- Recall that first order delays assume that the per-time-unit risk of transitions to the outflow remains equal throughout simulation (i.e. are memoryless)
- Problem: Often we know that transitions are *not* "memoryless” e.g.
  - It may be the transition reflects some physical delays not endogeneously represented (e.g. Slow-growth of bacterial)
  - Buildup of “damage” of high blood sugars (Glycosylation)

Higher Orders of Delays

- We can capture different levels of delay (with increasing levels of fidelity) using cascaded series of 1st order delays
- We call the delay resulting from such a series of *k* 1st order delays a "kth order delay"
  - E.g. 2 first order delays in series yield a 2nd order delay
- The behaviour of a kth order delay is a reflection of the behaviour of the 1st order delays out of which it is built
- To understand the behaviour of kth order delays, we will keep constant the mean time taken to transition across the entire set of all delays

Recall: Simple 1st Order Decay



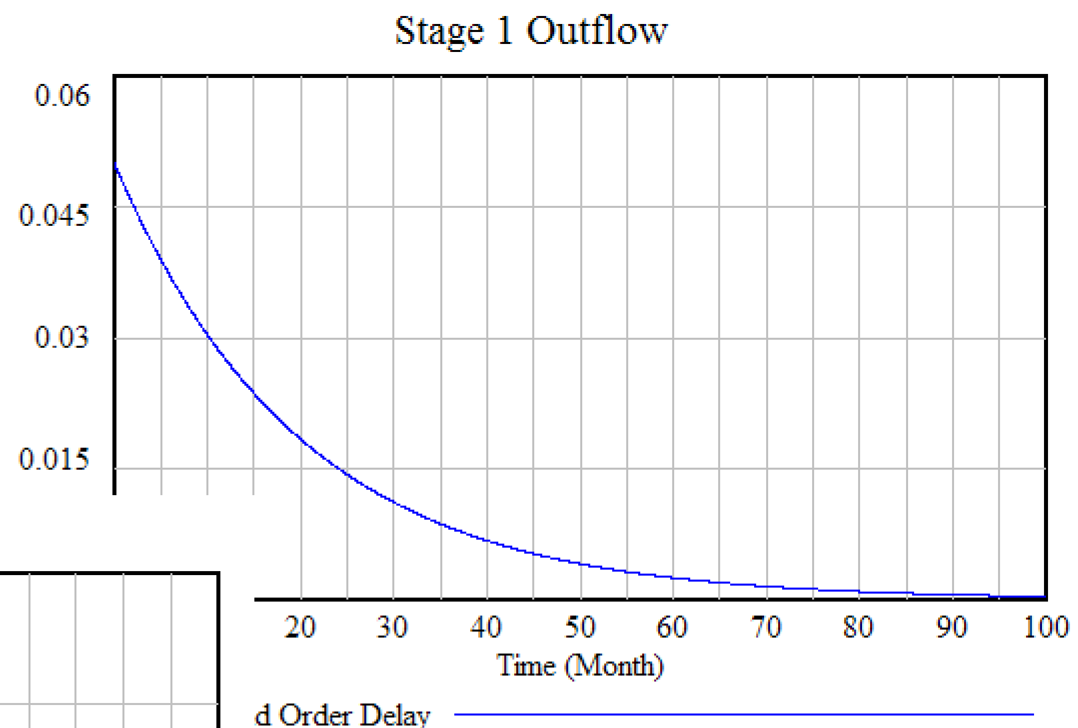
(Initial Value: 1)

Use Formula: People with Virulent Infection/Mean time until Death

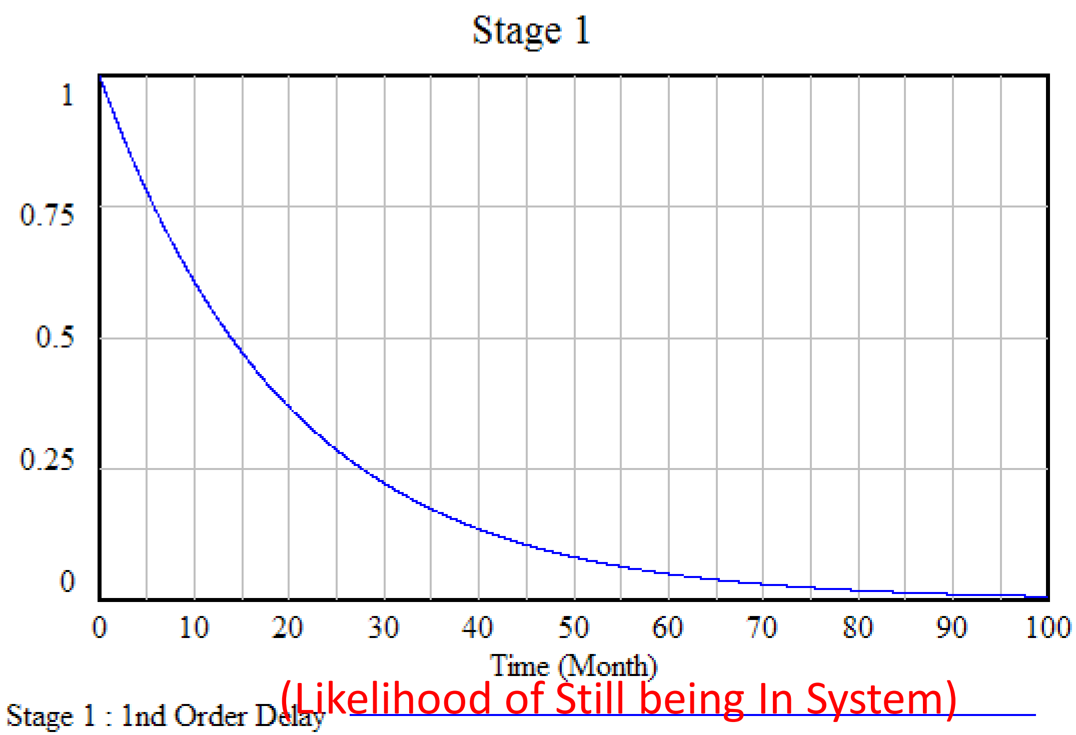
Recall: 1st Order Delay Behaviour

- ***Conditional*** transition prob: For a 1st Order delay, the per-time-unit likelihood of leaving *given that one has not yet left the stock* remains constant
- ***Unconditional*** transition prob: For a 1st Order delay, the unconditional per-time-unit likelihood of leaving declines exponentially
  - i.e. if were were originally in the stock, our chance of having left in the course of a given time unit (e.g. month) declines exponentially
    - This reflects the fact that there are fewer people who could still leave during this time unit!

Recall: 1^{st} Order Delay Behaviour

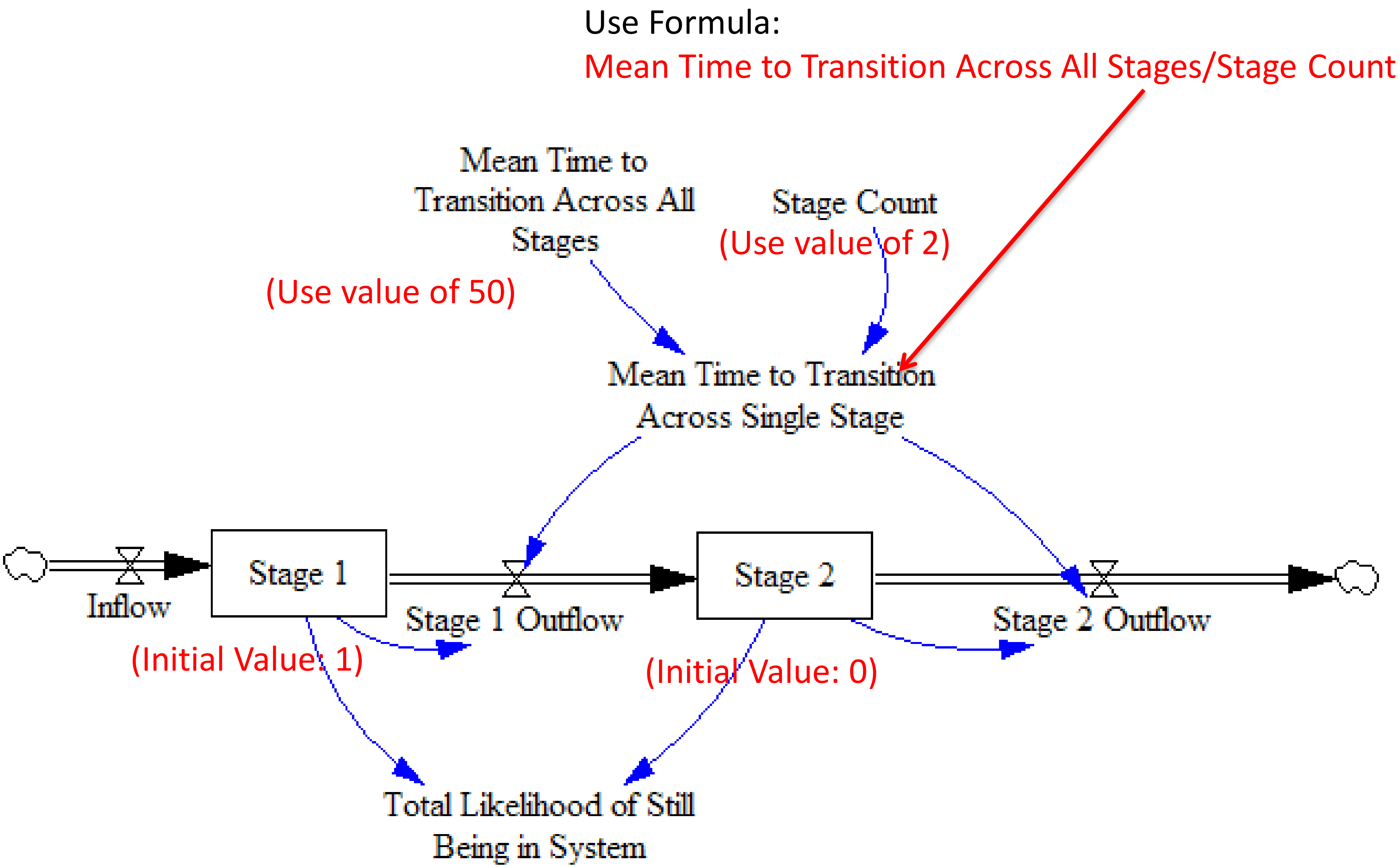


(Per-month chance of transitioning out during this month)

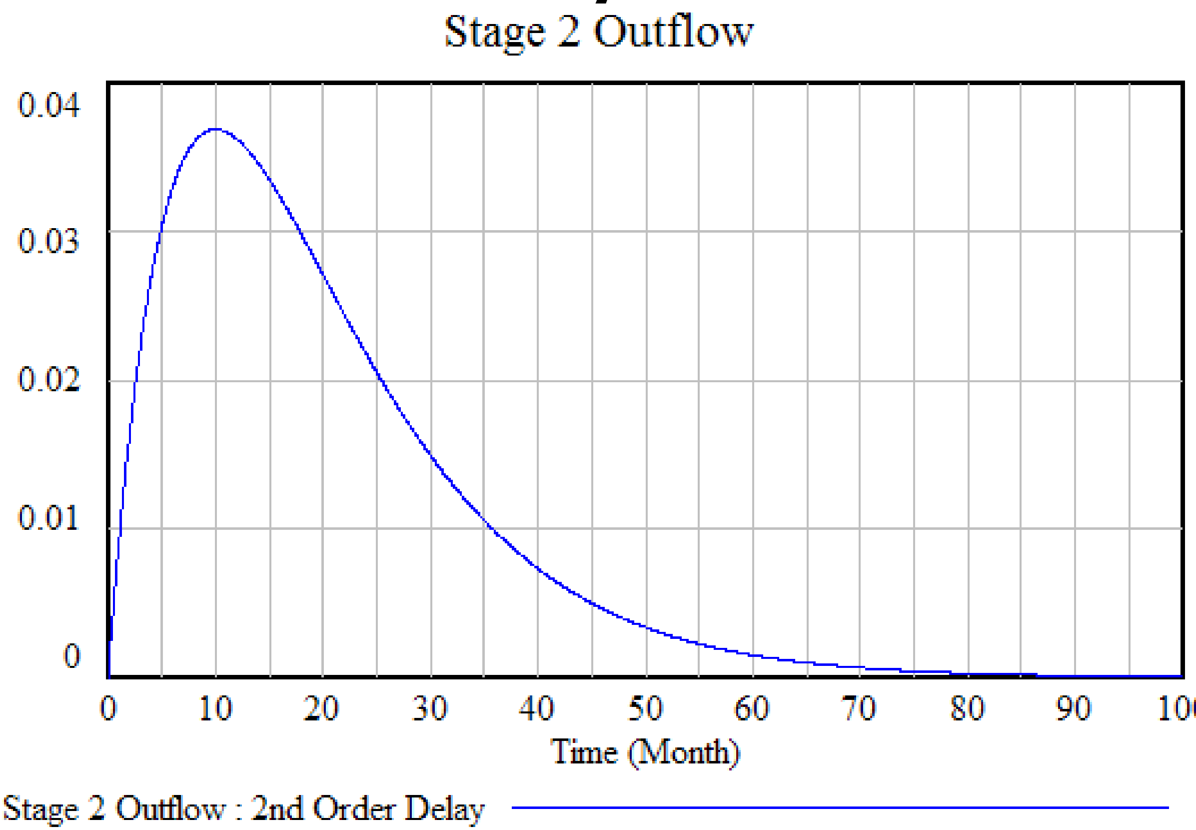


(Likelihood of Still being In System)

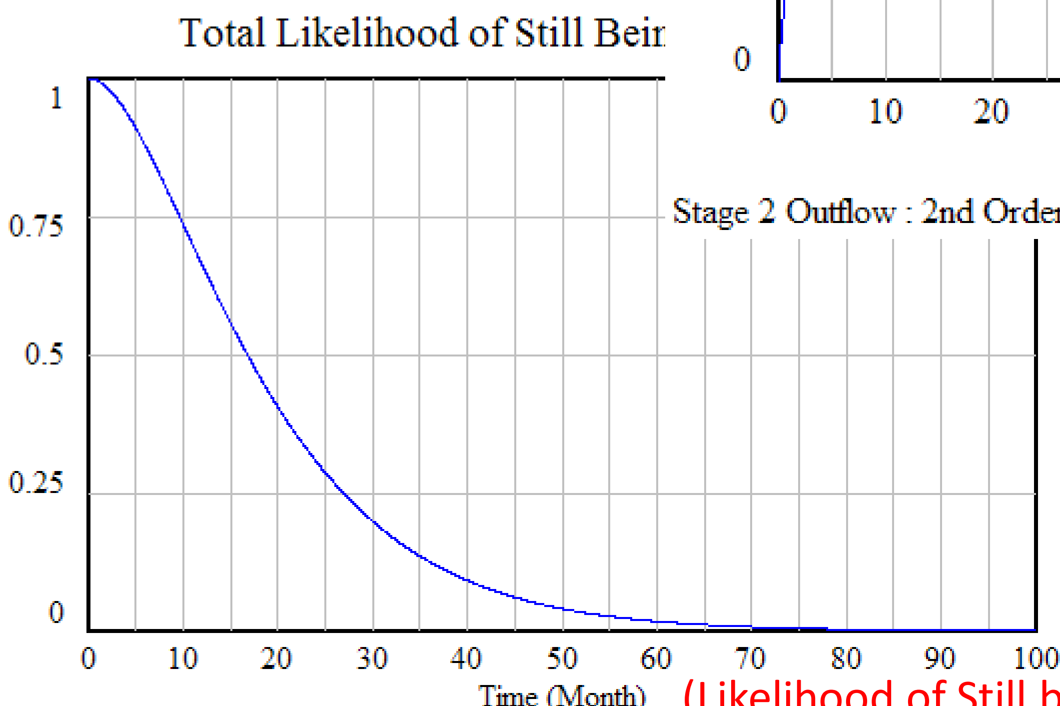
2^{nd} Order Delay



2nd Order Delay

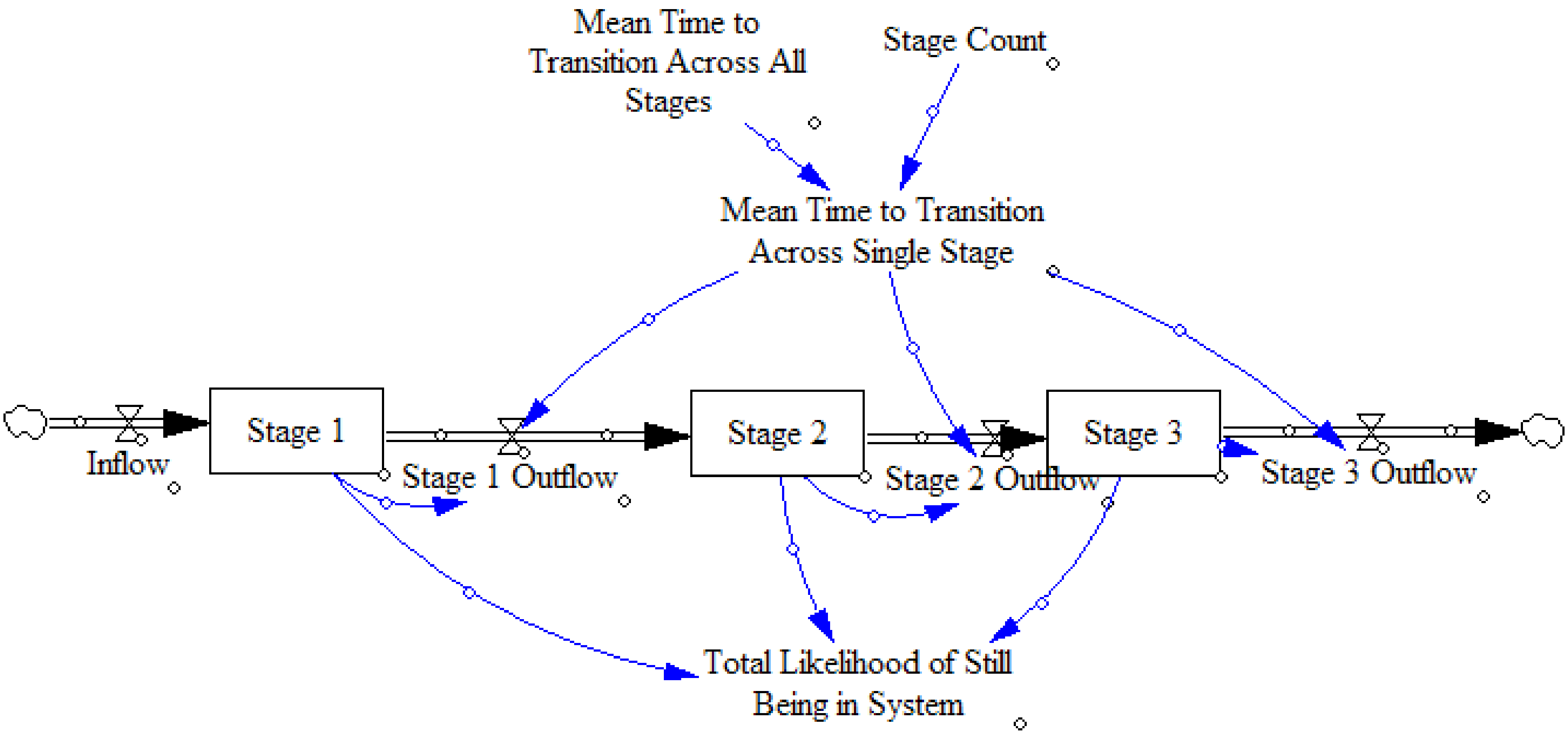


(Per-month chance of transitioning out during this month)

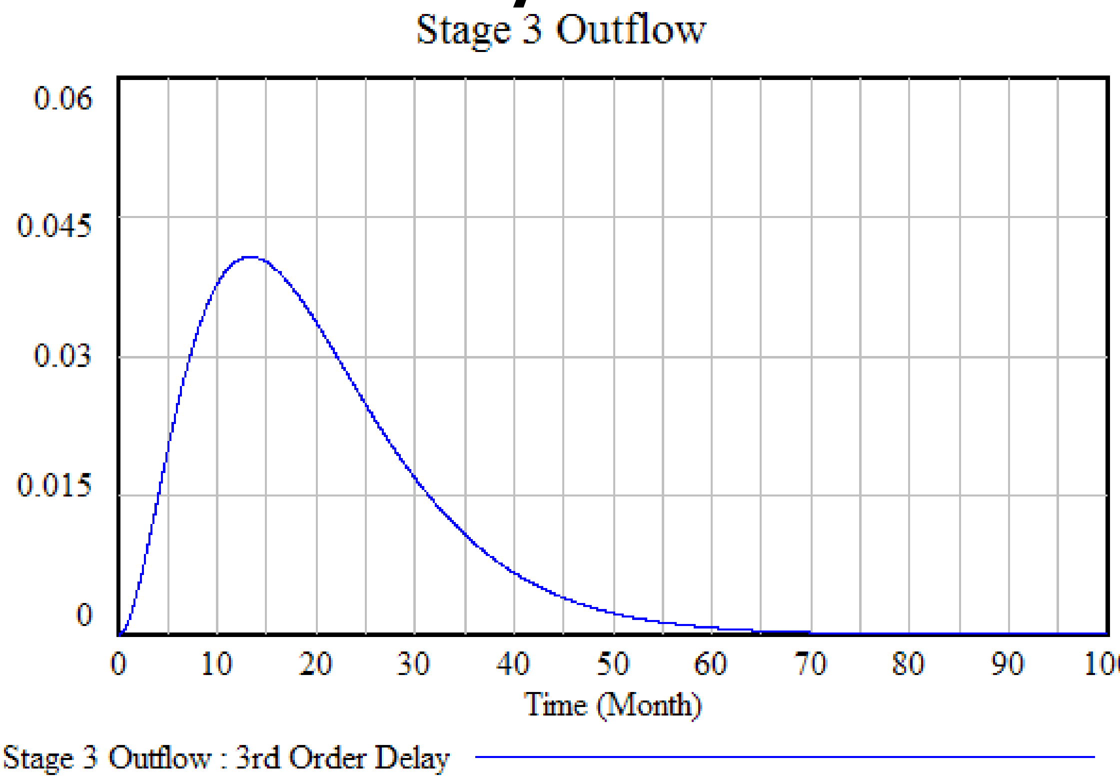


(Likelihood of Still being In System)

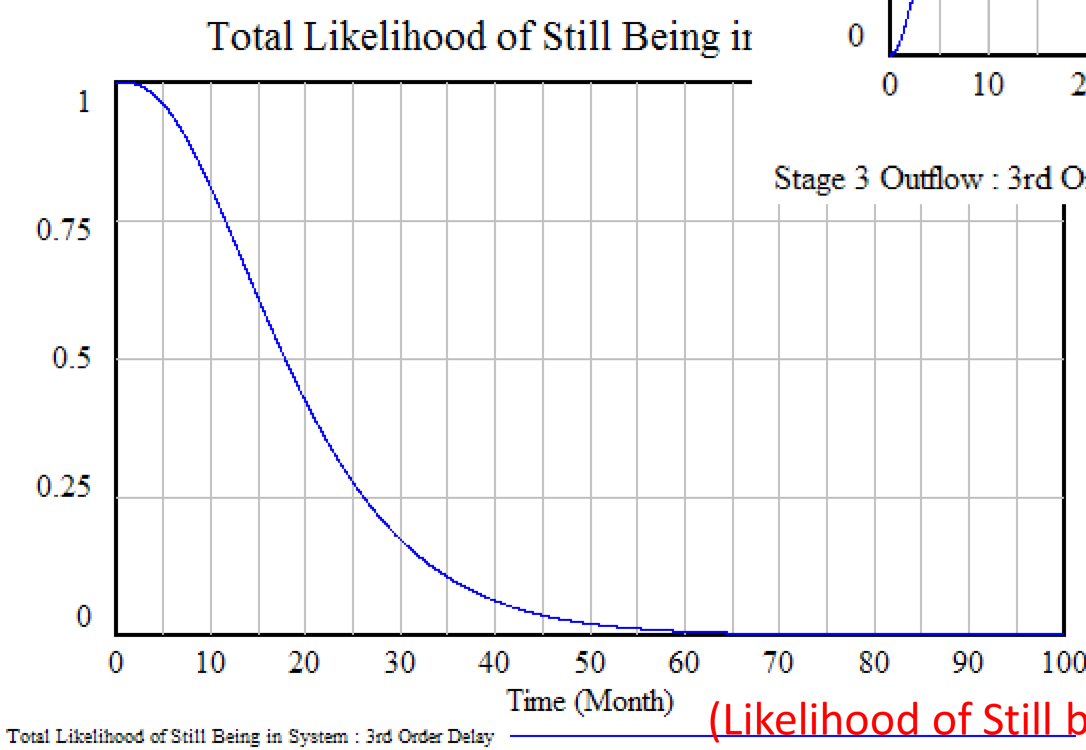
3rd Order Delay



3rd Order Delay

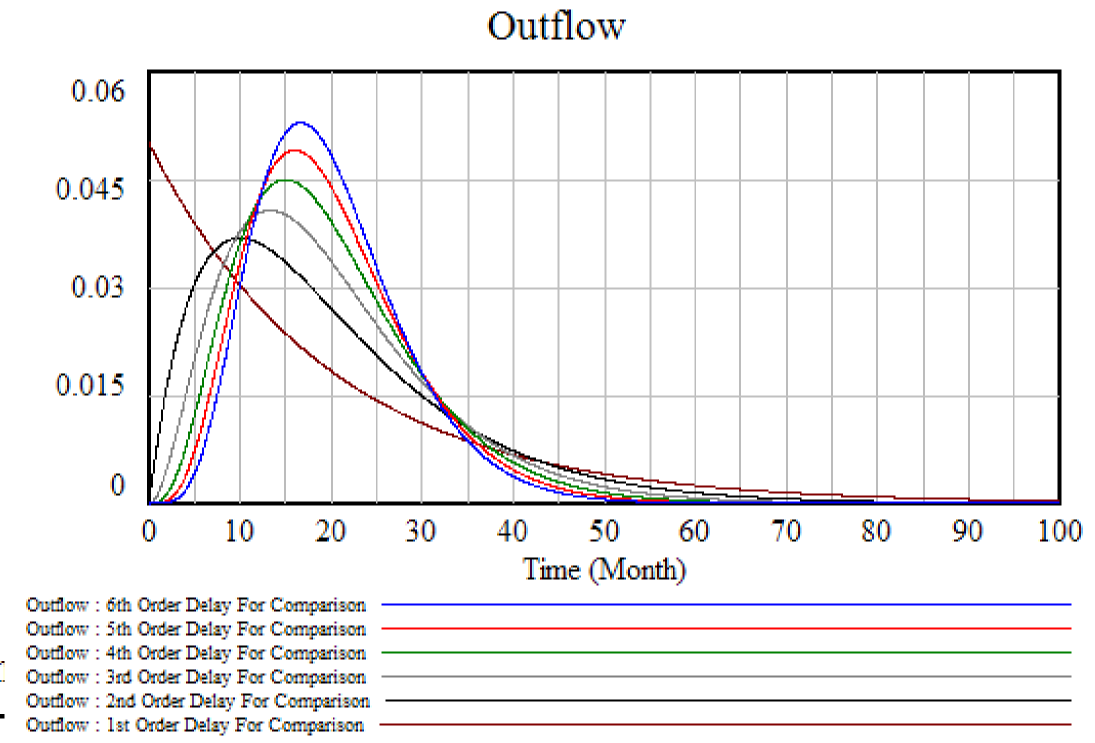


(Per-month chance of transitioning out during this month)

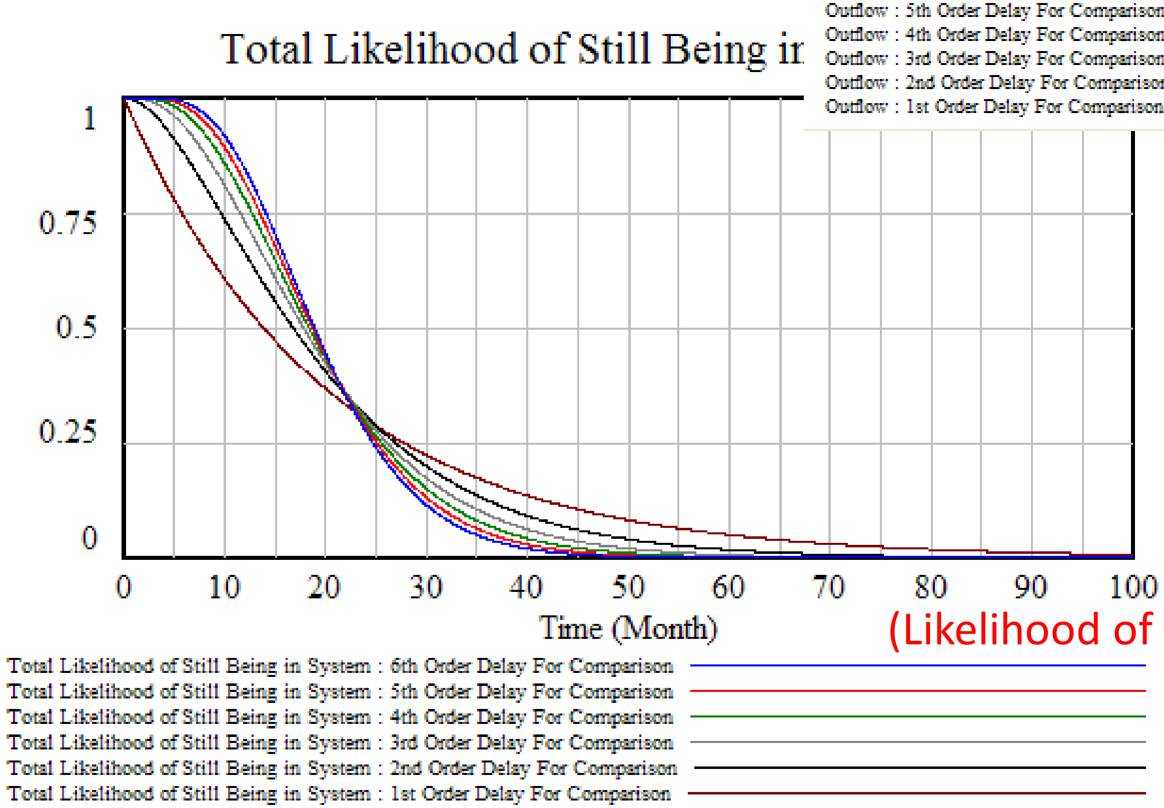


(Likelihood of Still being In System)

1^{st} through 6^{th} OrderDelays



(Per-month chance of transitioning out during this month)



(Likelihood of Still being In System)

Mean Times to Depart Final Stage

- Mean time of k stages is just k times mean time of one stage (e.g. if the mean time for leaving 1 stage requires time μ, mean time for k = k*μ
- In our examples, as we added stages, we reduced the mean time per stage so as to keep the total constant!
  - i.e. if we have k stages, the mean time to leave each stage is 1/k times what it would be with just 1 stage
- Infinite order delay: As we add more and more stages ($k \to \infty$), the distribution of time to leave the last stage approaches a normal distribution
  - If we reduce the mean time per stage so as to keep the total time constant, this will approach an impulse function
    - This indicates an exactly fixed time to transition through all stages!

Distribution of Time to Depart Final Stage

- The distributions for the total time taken to transition out of the last of k stages are members of the *Erlang*distribution family
  - These are the same as the distribution for the k^{th}interarrival time of a Poisson process
- k=1 gives exponential distribution (first order delay)
- As $k \rightarrow \infty$, approaches normal distribution (Gaussian pdf)

Parameters	$k > 0 \in \mathbb{Z}$ shape $\lambda > 0$ rate (real) alt.: $\theta = 1/\lambda > 0$ scale (real)
Support	$x \in [0; \infty)$
Probability density function (pdf)	$\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$
Cumulative distribution function (cdf)	$\frac{\gamma(k, \lambda x)}{(k-1)!} = 1 - \sum_{n=0}^{k-1} e^{-\lambda x} (\lambda x)^n / n!$
Mean	k/λ
Median	no simple closed form
Mode	$(k-1)/\lambda$ for $k \geq 1$
Variance	k/λ^2
Skewness	$\frac{2}{\sqrt{k}}$
Excess kurtosis	$\frac{6}{k}$
Entropy	$(1-k)\psi(k) + \ln \frac{\Gamma(k)}{\lambda} + k$
Moment-generating function (mgf)	$(1 - t/\lambda)^{-k}$ for $t < \lambda$
Characteristic function	$(1 - it/\lambda)^{-k}$

From Wikipedia, 2009

Notes

- We do not generally define k^{th}order delays simply as a means to the end of capturing a certain distribution
  - Often representing each stage for its own sake is desirable (see examples)
    - Different causal influences
  - Often we represent each such stage as a 1st order delay
- With that proviso, many modeling packages (including Vensim) directly support higher-order delays – use with caution

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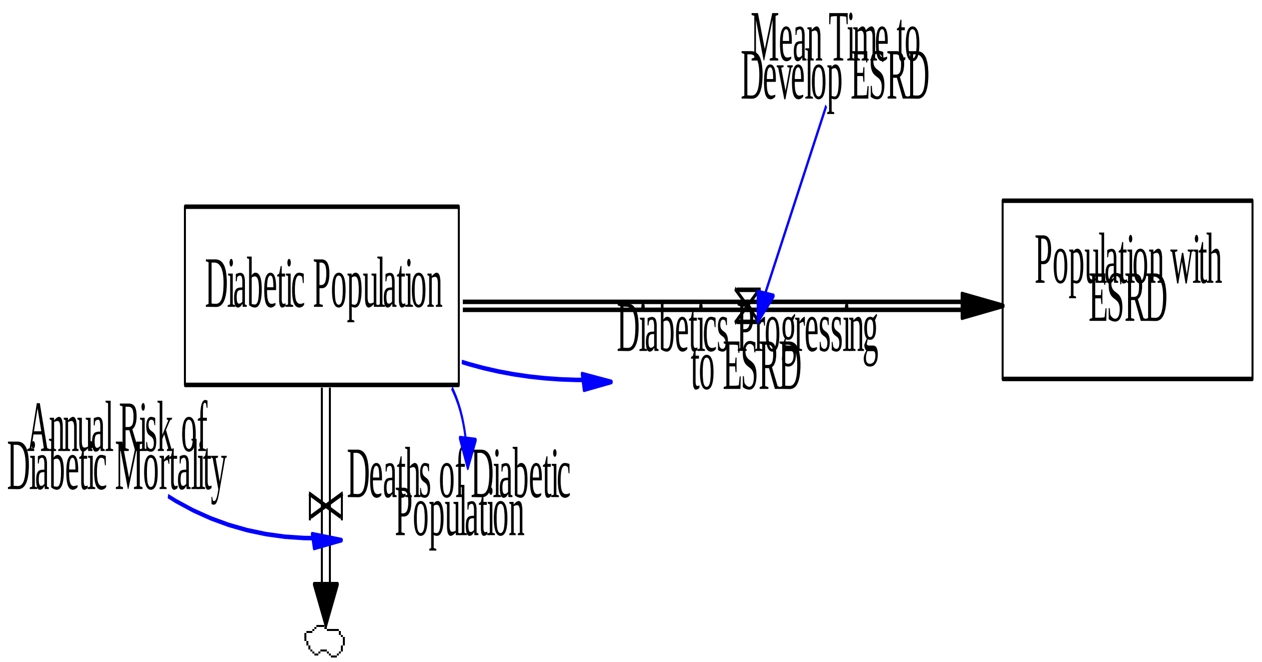
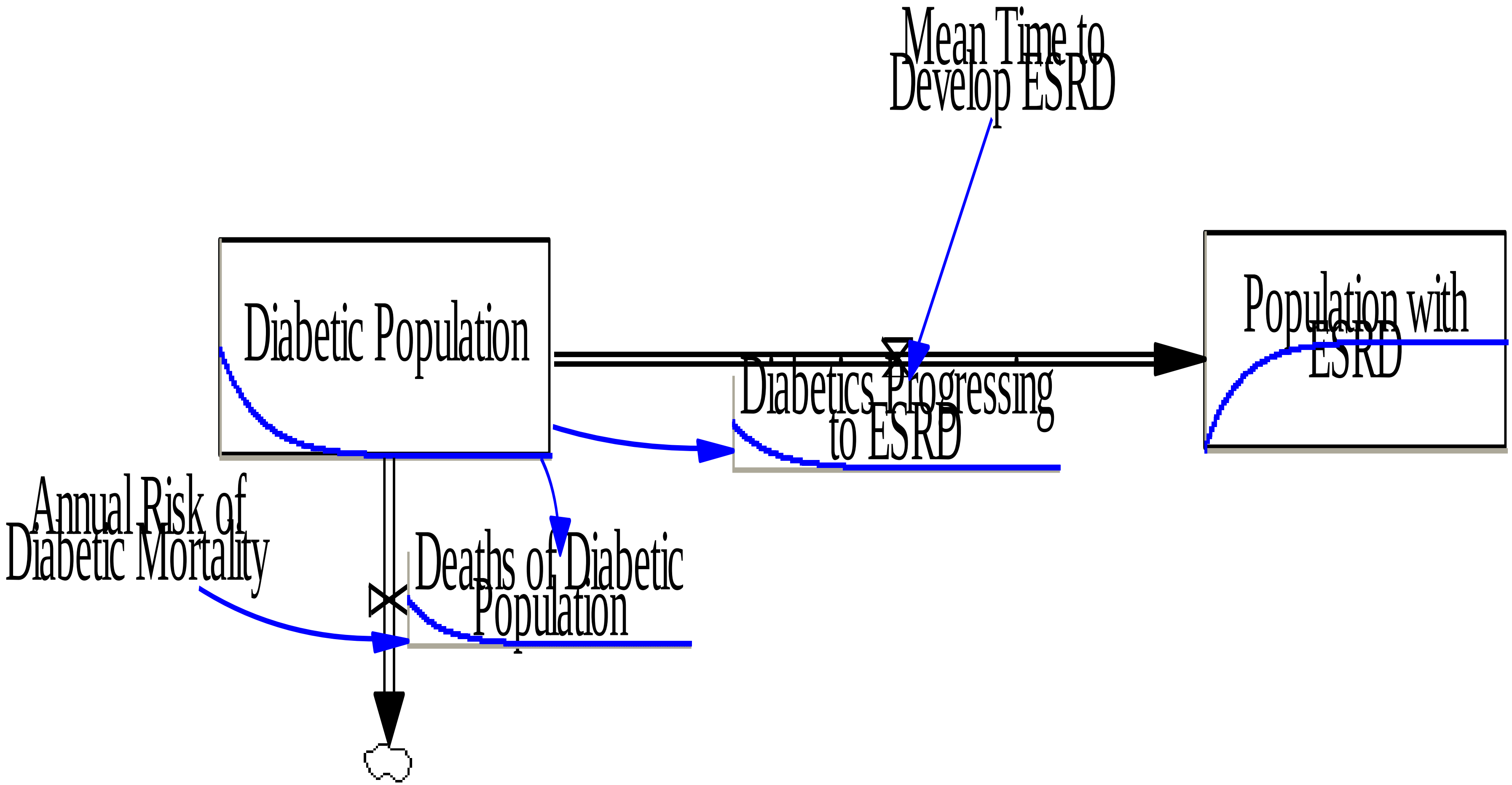
Delays & Competing Risks

Competing Risks

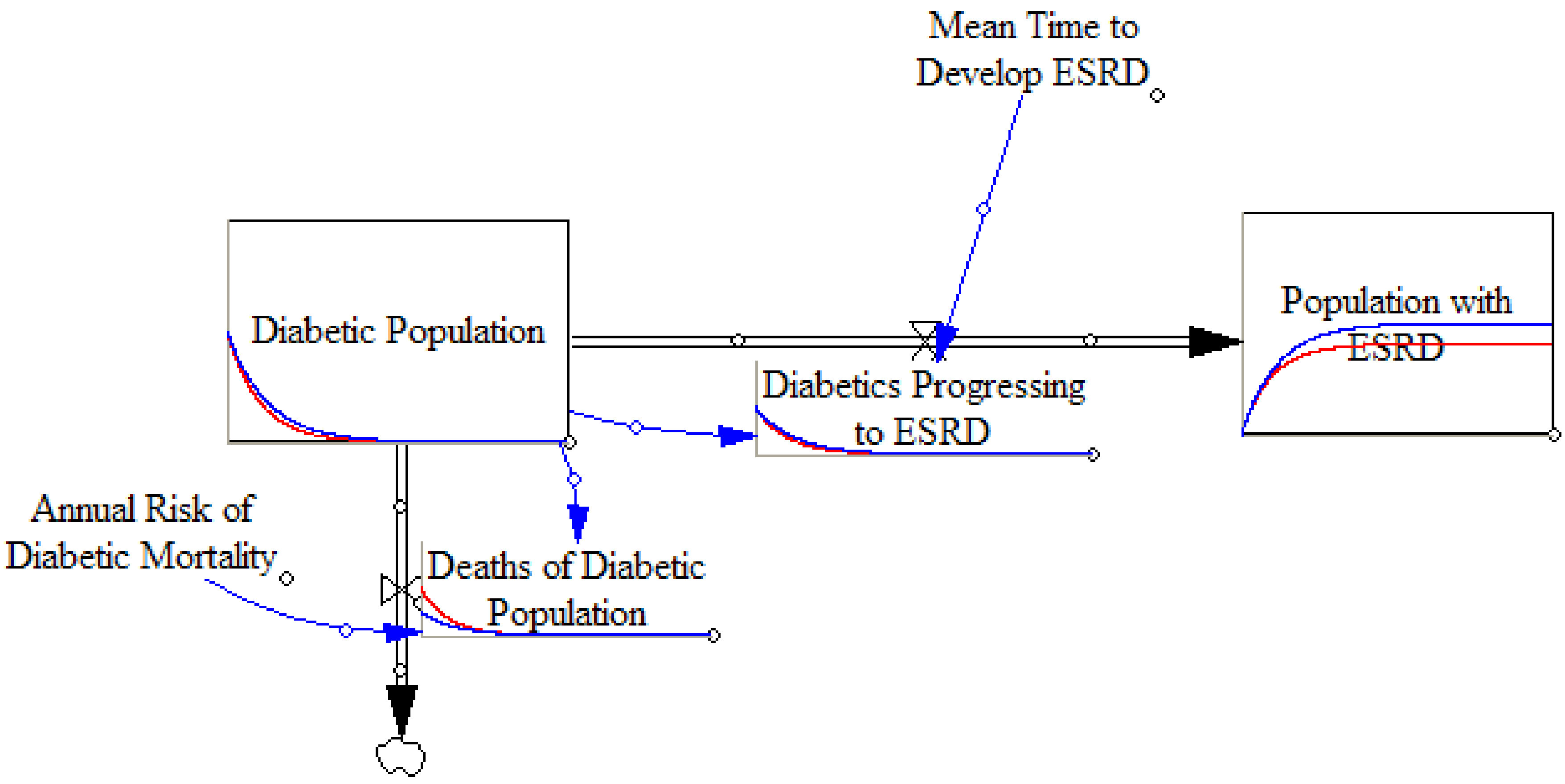
- Suppose we have another outflow from the stock. How does that change our mean time of proceeding specifically down flow 1 (here, developing diabetes)?



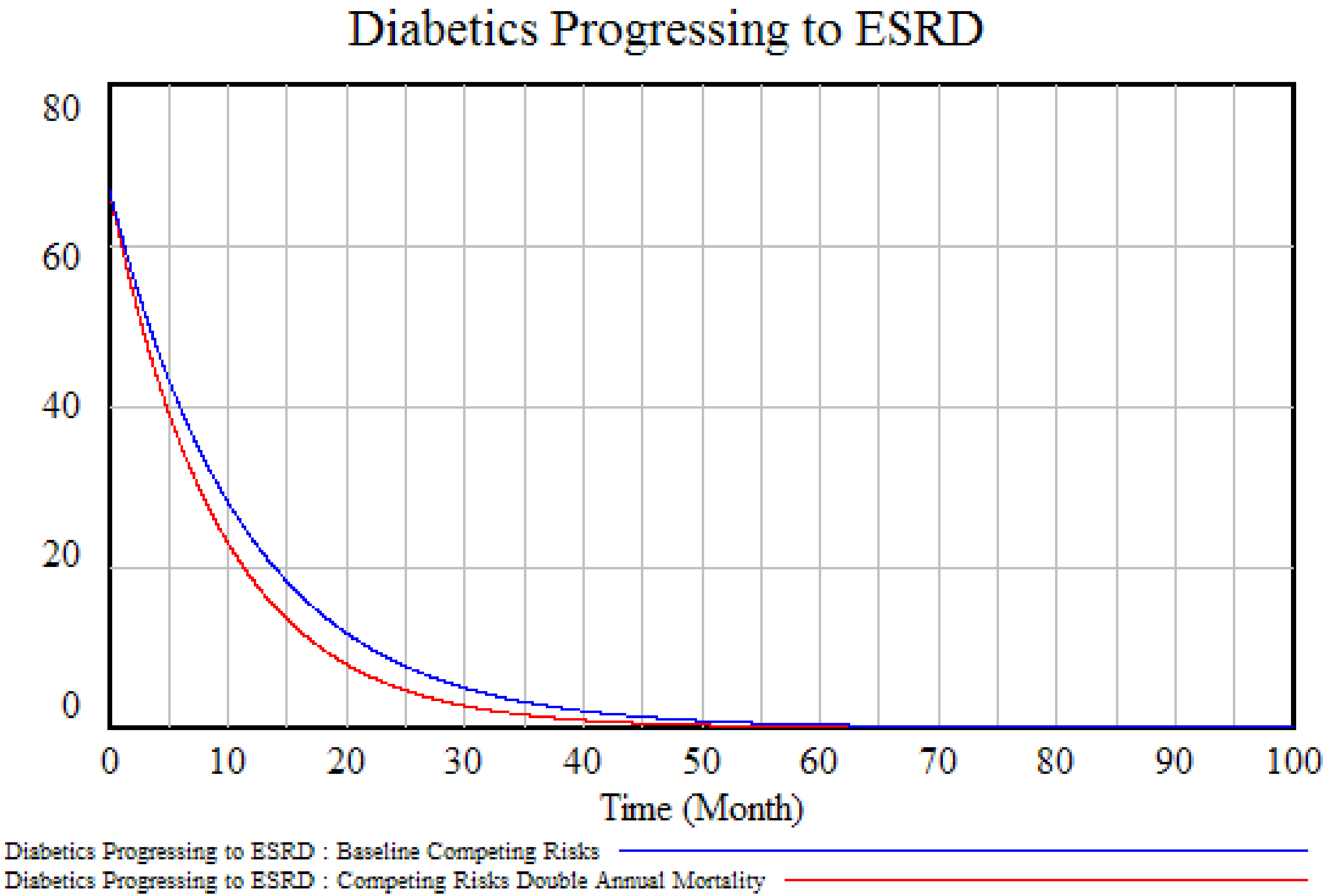
Basic Dynamics



Effect of Doubling Diabetic Mortality Rate



Effect on *Progression* Rates to ESRD



Do the two scenarios have the same or different *mean times to develop ESRD?* If different, which scenario is larger?

Why the Lower Mean Time?

- Why is the mean time to transition different, despite the fact that we didn't change the transition parameter?
- Mathematical explanation (Following slides): Calculation of mean time varies with mortality rate
- Intuition:
  - Higher death rate$\Rightarrow$Diabetic population will rapidly decrease &transitions to ESRD will be skewed towards earlier transitions$\Rightarrow$Earlier mean time to transition
  - Lower death rate$\Rightarrow$Diabetic population will decrease less rapidly & many will make later transitions to ESRD $\Rightarrow$Later mean time to transition

Competing Risks Stock Trajectory Solution Procedure

$$\frac{dx}{dt} = -\alpha x - \beta x = -(\alpha + \beta) x$$

- Suppose we start x at time 0 with initial value x(0), and we want to find the value of x at time T
- This is just like our previous differential equation, except that "α" has been replaced by "$(\alpha+\beta)$"
  - The solution must therefore be the same as before, with the appropriate replacement
  - Thus

$$x(T) = x(0)e^{-(\alpha+\beta)T}$$

Mean Time to Leave: Competing Risks

- p(t)dt here is the likelihood of a person leaving *via flow 1* (e.g. developing ESRD) exactly between time t &dt+t
  - We start the simulation at t=0, so p(t)=0 for t<0
  - For t>0, P(leaving on flow 1 exactly between time t &dt+t)=P(leaving on flow 1 exactly between time t &t+dt|Still have not left by time t)P(Still have not left by time t)

For T>0, P(Still have not left by time T)= $e^{-(\alpha+\beta)T}$

For P(leaving exactly between time t and t+dt|Still have not left by time t)

Recall: For us, probability of leaving in a time dt always=αdt

Thus P(leaving exactly between time t and t+dt|Still have not left by time t)= αdt

P(t)dt=P(leaving exact b.t. time t &dt+t)

$$=\alpha e^{-(\alpha+\beta)T}dt$$

Mean Time to Transition via Flow 1: Competing Risks

- By the same procedure as before, we have

$$E[p(t)] = \alpha \int_{t=0}^{t=\infty} te^{-(\alpha+\beta)T} dt$$

- Using the formula we derived for the integral expression, we have

$$E[p(t)] = \frac{\alpha}{(\alpha+\beta)^2}$$

- Note that this correctly approaches the single-flow case as $\beta \rightarrow 0$

“Aging Chains” (including successive 1st Order Delays & Competing Risks) in our Model of Chronic Kidney Disease